THE UNIVERSITY OF CALGARY

Programming with a Quantum Stack

by

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Abstract

This thesis presents the semantics of *quantum stacks* and a functional quantum programming language, L-QPL. An operational semantics for L-QPL based on quantum stacks in the form of a term logic is developed and used as an interpretation of quantum circuits. The operational semantics is then extended to handle recursion and algebraic datatypes. Recursion and datatypes are not concepts found in quantum circuits, but both are generally required for modern programming languages.

The language L-QPL is introduced in a discussion and example format. Various example programs using both classical and quantum algorithms are used to illustrate features of the language. Details of the language, including handling of qubits, general data types and classical data are covered.

The quantum stack machine is then presented. Supporting data for operation of the machine are introduced and the transitions induced by the machine's instructions are given.

Acknowledgments

Anyone who has written a thesis knows that an incredible number of people support and inspire the writer. I have been incredibly fortunate to have many gifted people in my life during this period.

My greatest appreciation to Dr. Robin Cockett for working with me these past years. His tireless efforts and indefatigable enthusiasm are a constant source of inspiration.

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Finally, a wonderful thanks to my sweetheart Marie, who supported me, inspired me and proofread the thesis. C'est pour toi que je suis.

Dedication

Dedicated to my wonderful wife Marie Gélinas Giles and to my grandchildren.

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Chapter 1

Introduction

This thesis introduces a *quantum stack* and its semantics together with a quantum programming language which has quantum control and classically controlled data types. The quantum stack gives the basis for an abstract (quantum) machine, the QSM. The quantum programming language, L-QPL, gives programmers a high-level functional language, suitable for experimination and design of quantum algorithms. This work was motivated by the desire to provide a semantically correct programming language for programming quantum algorithms at a higher level than bits and qubits.

1.1 Why a quantum programming language?

Currently, it is not clear that there will ever be a quantum computer having a number of qubits comparable to the number of bits available on classical computers. Thus, it is reasonable to ask what is the point of a quantum programming language. The compiler for L-QPL presented in this thesis targets a virtual machine — the quantum stack machine, which has been implemented on a classical computer. Any significant quantum algorithm written in L-QPL and simulated classically by this virtual machine is inefficient.

However, there *are* many reasons to create and use a quantum programming language and to understand how to organize a virtual machine for such a language.

Theory of algorithms. The current understanding of the limits of practical computability has led researchers and practitioners to consider probabilistic (and quantum) algorithms. This increases the understanding of those problems and sometimes may lead back to classical algorithms.

Quantum algorithms subsume probabilistic algorithms in that providing language support for quantum computing also gives probabilistic support. A simple example of this can be seen in the program to generate a coin toss given in figure 4.5 on page 72.

Quantum algorithm experimentation. Thinking about quantum algorithms is enhanced by having a high level way of expressing algorithms. A high level language such as L-QPL allows the researcher or practitioner to design an algorithm at an altogether different level than the standard bits and qubits of quantum circuits. An emulator and virtual machine as provided by this thesis allows experimentation with quantum algorithms, allowing a broader exploration of the field.

Quantum computer design. The thesis presents a novel view of quantum computation using a quantum stack machine. This suggests a way of organizing quantum computation with a quantum stack machine as the central element. This may stimulate others to consider how to realize such a machine with quantum devices.

1.2 Previous research in this area

For this thesis, both functional quantum computing languages and quantum simulators were considered as relevant prior research. Previous quantum languages were restricted only to functional languages due to the variety of differences between functional and imperative languages, especially the acceptance of type safety as a basic component of functional languages and the lack of global variables in such languages.

1.2.1 QPL by Peter Selinger

In 2002, Peter Selinger presented a description and categorical semantics for a functional quantum programming language in [Sel04].

Much of the work in this thesis was inspired by and often based upon the language described therein. Dr. Selinger first presented a diagrammatic language consisting of picture fragments corresponding to various operations in the language. In later sections of the paper, QPL and Block QPL are introduced. QPL closely mirrors the diagrammatic language with the addition of a few minor restrictions. Block QPL restricts the language further by creating a structured language that would allow allocation of data in a stack based environment rather than on a heap as required for QPL. The syntax for QPL is can be found in [Sel04].

An alternative direction in quantum programming languages, closely related to QPL, was explored in a series of papers ([AG05], [AGVS05], [GA05a], and [GA05b]) by Altenkirch and Grattage. L-QPL does not follow the direction taken by QML. The current compiler for QML translates QML code to Haskell, extended by types and functions as presented in [Sab03].

Comparison with and contrast to L-QPL

While inspired by QPL, L-QPL has diverged considerably. The syntax of L-QPL has been changed and extended, data construction has been added, quantum control by qubits and data types consisting of qubits in an integral part of the language and a demarcation between classical and quantum data is included in L-QPL.

The two languages are similar in that they both have qubits as a first class datatype and provide standard operations on qubits. The provided operations include a basis of unitary transformations and measurement.

Both languages provide procedures, although with different syntax. Each has an assignment statement. Each is a functional language in the sense that a statement of the language is a function from its inputs to its outputs.

The greatest difference between QPL and L-QPL is that QPL is a bit and qubit oriented language, while L-QPL was designed to work with algebraic data types. L-QPL retains qubits, but bits are not built-in to L-QPL.

The syntax for unitary transformations differs between the languages, with QPL using an operator-assignment type of syntax and L-QPL syntactically treating transformations in the same way as function calls. Quantum control in QPL is done via assuming various built-in controlled transforms, while L-QPL requires specifying the control variables of a transform explicitly.

QPL provides explicit looping based upon a bit's value. In L-QPL, all looping is done via recursion.

The semantics of L-QPL is inspired by QPL. We provide an operational semantics in this thesis. An explicit semantic comparison is left for future work.

1.2.2 Quantum simulators

For this thesis, I considered two Haskell based simulators and one in C++. The first Haskell simulator is from Shin-Cheng Mu and Richard Bird [MB01] in 2001 and the second was published by Amr Sabry [Sab03] in 2003. The C++ simulator is discussed in [BCS03], although the primary intent of this latter paper is to discuss a possible architecture for quantum computers. The C++ implementation, available on the internet, is a simulator and a set of classes intended to serve as an extension to the C++ language ([Str97]).

Each of these papers provide ways to represent qubits in their language of choice, typically by storing the coefficients of the standard dirac notation in some way. (i.e, if

$q = \alpha |0\rangle + \beta |1\rangle$ for some complex α and β, the α and β are stored).

The salient feature of these papers, from our perspective, is that they *simulate* rather than *emulate* the quantum computation. Simulate means that at measurments the simulator does a "throw the dice" and actually collapses qubits. Emulate means that the emulator continues to track all possible outcomes of the quantum computation. The quantum stack machine emulates a quantum computation by using the density matrix of the entire system.

This creates two major differences between these simulators and the quantum stack machine.

The first, a postive effect for the quantum stack machine, directly affects how an experimentor can view results. Consider Grover's search algorithm in a simple case, choosing 1 number out of 16. The probability of success in this case is 96.1%, provided the correct number of iterations of applying the grover transform is chosen. (See appendix C.2.1 on page 150 and [Wat] for details.) To verify this by experimentation in a simulator requires a large number of "runs" of the simulator. To verify that in the quantum stack machine, an emulator, requires a single "run".

The second effect is on the space used. In a simulator, space used is on the order of 2^N where N is the number of qubits. In the quantum stack machine emulator, the space used is on the order of 4^N, the square of the simulator's space usage. This issue is mitigated by various sparse storage techniques, but can still limit the usability of the emulator vs. the simulator when experimenting with larger numbers of qubits.

1.3 Description and reading guide

The two main contributions of this thesis, the quantum stack and L-QPL, may be reviewed independently of each other.

For those interested in reviewing the introduction of the quantum stack machine and its implementation the recommended reading path is the chapter on semantics, chapter 3, followed by chapter 5. In appendix B, further details are given on quantum stack machine instructions and the translation of L-QPL into quantum stack machine code.

Chapter 4 is the primary required reading for learning L-QPL. All facets of the language are presented in that chapter interspersed with a number of examples. Further examples of programs in L-QPL can be found in appendix C, including quantum arithmetic, Grover's search algorith, Simon's algorithm and order finding. A complete BNF description of L-QPL is available in appendix A.

Details of running the L-QPL compiler and the quantum stack machine emulator are given in appendix D.

Chapter 2

Quantum computation and circuits

2.1 Linear algebra

Quantum computation requires familiarity with the basics of linear algebra. This section will give definitions of the terms used throughout this thesis. Further information and details may be found in many university level algebra texts, e.g., [Lan02].

2.1.1 Basic definitions

The first definition needed is that of a *vector space*.

Definition 2.1.1 (Vector Space). *Given a field* F*, whose elements will be referred to as scalars, a* vector space *over* F *is a non-empty set* V *with two operations,* vector addition *and* scalar multiplication*.* Vector addition *is defined as* $+ : \mathsf{V} \times \mathsf{V} \to \mathsf{V}$ *and denoted as* $\boldsymbol{v} + \boldsymbol{w}$ *where* $\boldsymbol{v}, \boldsymbol{w} \in \mathsf{V}$*. The set* V *must be an abelian group under* $+$*.* Scalar multiplication *is defined as* $\ : \mathsf{F} \times \mathsf{V} \to \mathsf{V}$ *and denoted as* $c\boldsymbol{v}$ *where* $c \in \mathsf{F}, \boldsymbol{v} \in \mathsf{V}$*. Scalar multiplication distributes over both vector addition and scalar addition and is associative.* F*'s multiplicative identity is an identity for scalar multiplication.*

The specific algebraic requirements are:

1. $\forall \mathbf{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathsf{V},\ (\mathbf{u} + \boldsymbol{v}) + \boldsymbol{w} = \mathbf{u} + (\boldsymbol{v} + \boldsymbol{w})$;
2. $\forall \mathbf{u}, \boldsymbol{v} \in \mathsf{V},\ \mathbf{u} + \boldsymbol{v} = \boldsymbol{v} + \mathbf{u}$;
3. $\exists 0 \in \mathsf{V}$ such that $\forall \boldsymbol{v} \in \mathsf{V}, 0 + \boldsymbol{v} = \boldsymbol{v}$;
4. $\forall \mathbf{u} \in \mathsf{V}, \exists \boldsymbol{v} \in \mathsf{V}$ such that $\mathbf{u} + \boldsymbol{v} = 0$;

5. $\forall \mathbf{u}, \mathbf{v} \in V, c \in F,\ c(\mathbf{u}+\mathbf{v}) = c\mathbf{u} + c\mathbf{v}$;

6. $\forall \mathbf{u} \in V, c, d \in F,\ (c+d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$;

7. $\forall \mathbf{u} \in V, c, d \in F,\ (cd)\mathbf{u} = c(d\mathbf{u})$;

8. $\forall \mathbf{u} \in V,\ 1\mathbf{u} = \mathbf{u}$.

Examples of vector spaces over F are: $F^{n\times m}$ – the set of $n \times m$ matrices over F; and F^n – the $n-$fold Cartesian product of F. $F^{n\times 1}$, the set of $n \times 1$ matrices over F is also called the space of column vectors, while $F^{1\times n}$, the set of row vectors. Often, F^n is identified with $F^{n\times 1}$.

This thesis shall identify F^n with the column vector space over F.

Definition 2.1.2 (Linearly independent). *A subset of vectors $\{\mathbf{v}_i\}$ of the vector space V is said to be* linearly independent *when no finite linear combination of them, $\sum a_j \mathbf{v}_j$ equals $\mathbf{0}$ unless all the a_j are zero.*

Definition 2.1.3 (Basis). *A* basis *of a vector space V is a linearly independent subset of V that generates V. That is, any vector $u \in V$ is a linear combination of the basis vectors.*

2.1.2 Matrices

As mentioned above, the set of $n \times m$ matrices over a field is a vector space. Additionally, matrices compose and the tensor product of matrices is defined.

Matrix composition is defined as usual. That is, for $A = [a_{ij}] \in F^{m\times n}, B = [b_{jk}] \in F^{n\times p}$:

$$A\,B = \left[\left(\sum_j a_{ij} b_{jk}\right)_{ik}\right] \in F^{m\times p}.$$

Definition 2.1.4 (Diagonal matrix). *A* diagonal matrix *is a matrix where the only non-zero entries are those where the column index equals the row index.*

The diagonal matrix $n \times n$ with only $1'$s on the diagonal is the identity for matrix multiplication, and is designated by I_n.

Definition 2.1.5 (Transpose)**.** *The* transpose *of an* $n \times m$ *matrix* $A = [a_{ij}]$ *is an* $m \times n$ *matrix* A^t *with the* i, j *entry being* a_{ji}.

When the base field of a matrix is $\mathbb{C}$, the complex numbers, the *conjugate transpose* (also called the *adjoint*) of an $n \times m$ matrix $A = [a_{ij}]$ is defined as the $m \times n$ matrix A^* with the i, j entry being $\overline{a}_{ji}$, where $\overline{a}$ is the complex conjugate of $a \in \mathbb{C}$.

When working with column vectors over $\mathbb{C}$, note that $\mathbf{u} \in \mathbb{C}^n \implies \mathbf{u}^* \in \mathbb{C}^{1\times n}$ and that $\mathbf{u}^* \times \mathbf{u} \in \mathbb{C}^{1\times 1}$. This thesis will use the usual identification of $\mathbb{C}$ with $\mathbb{C}^{1\times 1}$. A column vector $\mathbf{u}$ is called a *unit vector* when $\mathbf{u}^* \times \mathbf{u} = 1$.

Definition 2.1.6 (Trace)**.** *The* trace, $Tr(A)$ *of a square matrix* $A = [a_{ij}]$ *is* $\sum a_{ii}$.

Tensor product

The tensor product of two matrices is the usual Kronecker product:

$$U \otimes V = \begin{bmatrix} u_{11}V & u_{12}V & \cdots & u_{1m}V \\ u_{21}V & u_{22}V & \cdots & u_{2m}V \\ \vdots & \vdots & \ddots & \\ u_{n1}V & u_{n2}V & \cdots & u_{nm}V \end{bmatrix} = \begin{bmatrix} u_{11}v_{11} & \cdots & u_{12}v_{11} & \cdots & u_{1m}v_{1q} \\ u_{11}v_{21} & \cdots & u_{12}v_{21} & \cdots & u_{1m}v_{2q} \\ \vdots & \vdots & \vdots & \ddots & \\ u_{n1}v_{p1} & \cdots & u_{n2}v_{p1} & \cdots & u_{nm}v_{pq} \end{bmatrix}$$

Special matrices

When working with quantum values certain types of matrices over the complex numbers are of special interest. These are:

Unitary Matrix : Any $n \times n$ matrix A with $AA^* = I\ (= A^*A)$.

Hermitian Matrix : Any $n \times n$ matrix A with $A = A^*$.

Positive Matrix : Any Hermitian matrix A in $\mathbb{C}^{n\times n}$ where $\mathbf{u}^* A\mathbf{u} \geqslant 0$ for all vectors $\mathbf{u} \in \mathbb{C}^n$. Note that for any Hermitian matrix A and vector u, $\mathbf{u}^* A\mathbf{u}$ is real.

Completely Positive Matrix : Any positive matrix A in $\mathbb{C}^{n\times n}$ where $I_m \otimes A$ is positive.

The matrix

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

is an example of a matrix that is *unitary*, *Hermitian*, *positive* and *completely positive*.

Superoperators

A *Superoperator* S is a matrix over $\mathbb{C}$ with the following restrictions:

1. S is *completely positive*. This implies that S is positive as well.

2. For all positive matrices A, $\mathrm{Tr}(S\,A) \leqslant \mathrm{Tr}(A)$.

2.2 Basic quantum computation

This section provides a basic introduction to the mathematical descriptions of quantum bits and quantum computations. For a more complete treatment of the subject, please see [NC00] or [Wat].

2.2.1 Quantum bits

Quantum computation deals with operations on qubits. A qubit is typically represented in the literature on quantum computation as a complex linear combination of $|0\rangle$ and $|1\rangle$, respectively identified with (1,0) and (0,1) in $\mathbb{C}^2$. Because of the identification of the basis vectors, any qubit can be identified with a non-zero vector in $\mathbb{C}^2$. In standard quantum computation, the important piece of information in a qubit is its direction rather than amplitude. In other words, given $q = \alpha |0\rangle + \beta |1\rangle$ and $q' = \alpha' |0\rangle + \beta' |1\rangle$ where $\alpha = \gamma\alpha'$ and $\beta = \gamma\beta'$, then q and q' represent the same quantum state.

A qubit that has either α or β zero is said to be in a *classical state*. Any other combination of values is said to be a *superposition*.

Section 2.3 on page 15 will introduce quantum circuits which act on qubits. This section will have some forward references to circuits to illustrate points introduced here.

2.2.2 Quantum entanglement

Consider what happens when working with a pair of qubits, p and q. This can be considered as the a vector in $\mathbb{C}^4$ and written as

$$\alpha_{00} |00\rangle + \alpha_{01} |01\rangle + \alpha_{10} |10\rangle + \alpha_{11} |11\rangle . \tag{2.1}$$

In the case where p and q are two independent qubits, with $p = \alpha |0\rangle + \beta |1\rangle$ and $q = \gamma |0\rangle + \delta |1\rangle$,

$$p \otimes q = \alpha\gamma |00\rangle + \alpha\delta |01\rangle + \beta\gamma |10\rangle + \beta\delta |11\rangle \tag{2.2}$$

where $p \otimes q$ is the standard tensor product of p and q regarded as vectors. There are states of two qubits that cannot be written as a tensor product. As an example, the state

$$\frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle \tag{2.3}$$

is not a tensor product of two distinct qubits. In this case the two qubits are said to be *entangled*.

2.2.3 Quantum gates

Quantum gates operate on qubits. These gates are conceptually similar to logic gates in the classical world. In the classical world the only non-trivial single bit gate is the Not gate which sends 0 to 1 and 1 to 0. However, there are infinitely many non-trivial quantum gates.

An $n-\mathtt{qubit}$ quantum gate is represented by a $2^n \times 2^n$ matrix. A necessary and sufficient condition for such a matrix to be a quantum gate is that it is *unitary*.

The entanglement of two qubits, p and q, is accomplished by applying a Hadamard transformation to p followed by a Not applied to q controlled by p. The circuit in figure 2.2 on page 16 shows how to entangle two qubits that start with an initial state of $|00\rangle$. See figure C.1 on page 145 for how this can be done in L-QPL.

A list of some common gates, together with their usual quantum circuit representation is given in the next section in table 2.1 on page 17.

2.2.4 Measurement

The other allowed operation on a qubit or group of qubits is measurement. When a qubit is measured it assumes only one of two possible values, either $|0\rangle$ or $|1\rangle$. Given

$$q = \alpha\,|0\rangle + \beta\,|1\rangle \tag{2.4}$$

where $|\alpha|^2 + |\beta|^2 = 1$, then measuring q will result in $|0\rangle$ with probability $|\alpha|^2$ and $|1\rangle$ with probability $|\beta|^2$. Once a qubit is measured, re-measuring will always produce the same value.

In multi-qubit systems the order of measurement does not matter. If p and q are as in equation (2.1) on the preceding page, let us suppose measuring p gives $|0\rangle$. The measure will result in that value with probability $|\alpha_{00}|^2 + |\alpha_{01}|^2$, after which the system collapses to the state:

$$\alpha_{00}\,|00\rangle + \alpha_{01}\,|01\rangle \tag{2.5}$$

Measuring the second qubit, q, will give $|0\rangle$ with probability $|\alpha_{00}|^2$ or $|1\rangle$ with probability $|\alpha_{01}|^2$.

Conversely, if q was measured first and gave us $|0\rangle$ (with a probability of $|\alpha_{00}|^2 + |\alpha_{10}|^2$) and then p was measured, p will give us $|0\rangle$ with probability $|\alpha_{00}|^2$ or $|1\rangle$ with probability $|\alpha_{10}|^2$.

Thus, when measuring both p and q, the probability of getting $|0\rangle$ from both measures is $|\alpha_{00}|^2$, regardless of which qubit is measured first.

Considering states such as in equation (2.3), measuring either qubit would actually force the other qubit to the same value. This type of entanglement is used in many quantum algorithms such as quantum teleportation.

2.2.5 Mixed states

The notion of *mixed states* refers to an outside observer's knowledge of the state of a quantum system. Consider a 1 qubit system

$$v = \alpha\,|0\rangle + \beta\,|1\rangle\,. \tag{2.6}$$

If v is measured but the results of the measurement are not examined, the state of the system is either $|0\rangle$ or $|1\rangle$ and is no longer in a superposition. This type of state is written as:

$$v = |\alpha|^2\{|0\rangle\} + |\beta|^2\{|1\rangle\}. \tag{2.7}$$

An external (to the state) observer knows that the state of v is as expressed in equation (2.7). Since the results of the measurement were not examined, the exact state (0 or 1) is unknown. Instead, a probability is assigned as expressed in the equation.

Thus, if the qubit ν is measured and the results are not examined, ν can be treated as a probabilistic bit rather than a qubit.

2.2.6 Density matrix notation

The state of any quantum system of qubits may be represented via a *density matrix*. In this notation, given a qubit ν, the coefficients of $|0\rangle$ and $|1\rangle$ form a column vector u. Then the density matrix corresponding to ν is uu^*. If $\nu = \alpha\,|0\rangle + \beta\,|1\rangle$,

$$\nu = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}\begin{pmatrix}\overline{\alpha} & \overline{\beta}\end{pmatrix} = \begin{pmatrix}\alpha\overline{\alpha} & \alpha\overline{\beta}\\ \beta\overline{\alpha} & \beta\overline{\beta}\end{pmatrix}. \tag{2.8}$$

When working with mixed states the density matrix of each component of the mixed state is added. For example, the mixed state shown in equation (2.7) on the preceding page would be represented by the density matrix

$$|\alpha|^2\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + |\beta|^2\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}|\alpha|^2 & 0\\ 0 & |\beta|^2\end{pmatrix}. \tag{2.9}$$

Note that since the density matrix of mixed states is a linear combination of other density matrices, it is possible to have two different mixed states represented by the same density matrix.

The advantage of this notation is that it becomes much more compact for mixed state systems. Additionally, scaling issues are handled by insisting the density matrix has a trace = 1. During a general quantum computation, as we shall see, the trace can actually fall below 1 indicating that the computation is not everywhere total.

Gates and density matrices

When considering a qubit q as a column vector and a unitary transform T as a matrix, the result of applying the transform T to q is the new vector Tq. The density matrix of the original qubit is given by $q\,q^*$, while the density matrix of the transformed qubit is $(Tq)(Tq)^*$, which equals $T(qq^*)T^*$. Thus, when a qubit q is represented by a density matrix A, the formula for applying the transform T to q is TAT^*.

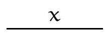
2.3 Quantum circuits

2.3.1 Contents of quantum circuits

Currently a majority of quantum algorithms are defined and documented using *quantum circuits*. These are wire-type diagrams with a series of qubits input on the left of the diagram and output on the right. Various graphical elements are used to describe quantum gates, measurement, control and classical bits.

Gates and qubits

The simplest circuit is a single wire with no action:



The next simplest circuit is one qubit and one gate. The qubit is represented by a single wire, while the gate is represented by a box with its name, G, inside it. This is shown in the circuit in figure 2.1. In general, the name of the wire which is input to the gate G may be different from the name of G's output wire. Circuit diagrams may also contain constant components as input to gates as in the circuit in figure 2.3 on the following page.

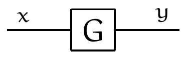


Figure 2.1: Simple single gate circuit

Future diagrams will drop the wire labels except when they are important to the concept under discussion.

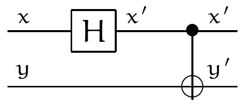


Figure 2.2: Entangling two qubits.

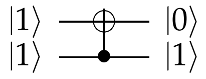


Figure 2.3: Controlled-Not of $|1\rangle$ and $|1\rangle$

Controlled gates, where the gate action depends upon another qubit, are shown by attaching a wire between the wire of the control qubit and the controlled gate. The circuit in figure 2.2 shows two qubits, where a Hadamard is applied to the top qubit, followed by a controlled-Not applied to the second qubit. In circuits, the control qubit is on the vertical wire with the solid dot. This is then connected via a horizontal wire to the gate being controlled.

A list of common gates, their circuits and corresponding matrices is given in table 2.1 on the following page.

Measurement

Measurement is used to transform the quantum data to classical data so that it may be then used in classical computing (e.g. for output). The act of measurement is placed at the last part of the quantum algorithm in many circuit diagrams and is sometimes just implicitly considered to be there.

While there are multiple notations used for measurement in quantum circuit diagrams, this thesis will standardize on the *D-box* style of measurement as shown in figure 2.4.

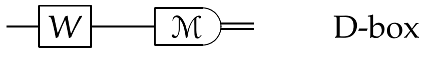


Figure 2.4: Measure notation in quantum circuits

A measurement may have a double line leaving it, signifying a bit, or nothing,

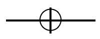
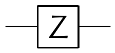
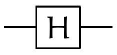
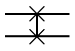
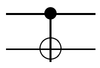
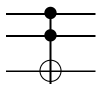
Gate	**Circuit**	**Matrix**
Not (X)		$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Z		$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Hadamard		$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
Swap		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
Controlled-Not		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
Toffoli		$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

Table 2.1: Gates, circuit notation and matrices

signifying a destructive measurement.

Operations affecting multiple qubits at the same time are shown by extending the gate or measure box to encompass all desired wires. In the circuit in figure 2.5, the gate U applies to all of the first three qubits and the measurement applies to the first two qubits.

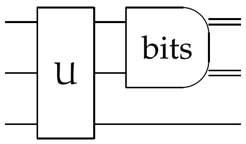


Figure 2.5: Examples of multi-qubit gates and measures

0-control and control by bits

The examples above have only shown control based upon a qubit being $|1\rangle$. Circuits also allow control on a qubit being $|0\rangle$ and upon classical values. The circuit in figure 2.6 with four qubits $(r_1, r_2, p$ and $q)$ illustrates all these forms of control.

At g_1, a Hadamard is 1—controlled by r_2 and is applied to each of r_1 and p. This is followed in column g_2 with the Not transform applied to r_2 being 0—controlled by r_1 . In the same column, a Z gate is 0—controlled by q and applied to p. p and q are then measured in column g_3 and their corresponding classical values are used for control in g_4. In g_4, the U_R gate is applied to both r_1 and r_2, but only when the measure result of p is 0 and the measure result of q is 1.

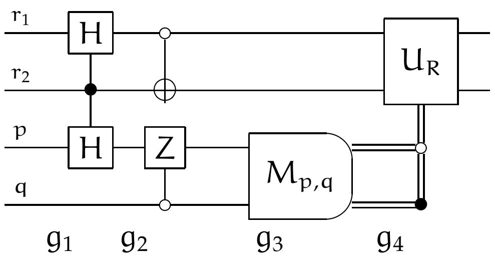


Figure 2.6: Other forms of control for gates

Multi-qubit lines

It is common to represent multiple qubits on one line. A gate applied to a multi-qubit line must be a tensor product of gates of the correct dimensions. The circuit in figure 2.7 shows n qubits on one line with the Hadamard gate (tensored with itself n times) applied to all of them. That is followed by a unary gate U_R tensored with $I^{\otimes(n-2)}$ and tensored with itself again. This will have the effect of applying an U_R gate to the first and last qubits on the line.

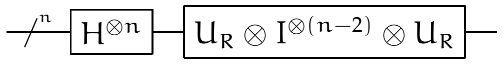


Figure 2.7: n qubits on one line

Other common circuit symbols

Two other symbols that are regularly used are the swap and controlled-Z, shown in the circuit in figure 2.8. Note that swap is equivalent to a series of three controlled-Not gates with the control qubit changing. This can also be seen directly by multiplying the matrices for the controlled-Not gates as shown in equation (2.10).

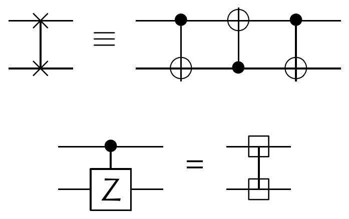


Figure 2.8: Swap and controlled-Z

$$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}. \tag{2.10}$$

2.3.2 Syntax of quantum circuits

Quantum circuits were originally introduced by David Deutsch in [Deu89]. He extended the idea of standard classical based gate diagrams to encompass the quantum cases. In his paper, he introduced the concepts of quantum gates, sources of bits, sinks and universal gates. One interesting point of the original definition is that it *does* allow loops. Currently, the general practice is not to allow loops of qubits. The commonly used elements of a circuit are summarized in table 2.2 on the following page.

A valid quantum circuit must follow certain restrictions. As physics requires qubits must not be duplicated, circuits must enforce this rule. Therefore, three restrictions in circuits are the *no fan-out*, *no fan-in* and *no loops* rules. These conditions are a way to express the *linearity* of quantum algorithms. Variables (wires) may not be duplicated, may not be destroyed without a specific operation and may not be amalgamated.

2.3.3 Examples of quantum circuits

This section will present three quantum algorithms and the associated circuits. Each of the circuits presented may be found in [NC00].

First, *quantum teleportation*, an algorithm which sends a quantum bit across a distance via the exchange of two classical bits. This is followed by the *Deutsch-Jozsa algorithm*, which provides information about the global nature of a function with less work than a classical deterministic algorithm can. The third example includes circuits for the *quantum Fourier transformation* and its inverse.

Quantum teleportation

The standard presentation of this algorithm involves two participants Alice and Bob. In the diagrams, these will be labelled as A and B. Alice and Bob first initialize two qubits to $|00\rangle$, then place them into what is known as an *EPR* (for Einstein, Podolsky

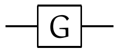
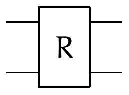
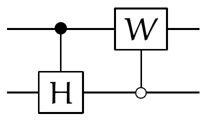
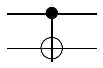
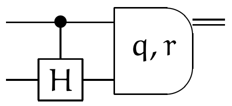
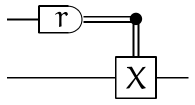
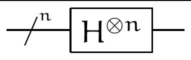
Desired element	**Element in a quantum circuit diagram.**	**Example**
qubit	A single horizontal line.	—
Classical bit	A double horizontal line.	=
Single-qubit gates	A box with the gate name (G) inside it, one wire attached on its left and one wire attached on the right.	
Multi-qubit gates	A box with the gate name (R) inside it, n wires on the left side and the same number of wires on the right.	
Controlled qubit gates	A box with the gate name (H, W) inside, with a solid (1-control) or open (0-control) dot on the control wire with a vertical wire between the dot and the second gate.	
Controlled-Not gates	A *target* $\oplus$, with a solid (1-control) or open (0-control) dot on the control wire with a vertical wire between the dot and the gate.	
Measurement	A *D-box* shaped node with optional names or comments inside. One to n single wires are attached on the left (qubits coming in) and 0 to n classical bit wires on the left. Classical bits may be dropped as desired.	
Classical control	Control bullets attached to horizontal classical wires, with vertical classical wires attached to the controlled gate.	
Multiple qubits	Annotate the line with the number of qubits and use tensors on gates.	

Table 2.2: Syntactic elements of quantum circuit diagrams

and Rosen) state. This is accomplished by first applying the Hadamard gate to Alice's qubit, followed by a controlled-Not to the pair of qubits controlled by Alice's qubit.

Then, Bob travels somewhere distant from Alice, taking his qubit with him[1].

At this point, Alice receives a qubit, ν, in an unknown state and has to pass ν on to Bob. She then uses ν as the control and applies a controlled-Not transform to this new pair. Alice then applies a Hadamard transform applied to ν.

Alice now measures the two qubits and sends the resulting two bits of classical information to Bob.

Bob then examines the two bits that he receives from Alice. If the bit resulting from measuring Alice's original bit is 1, he applies the Not (also referred to as X) gate ($= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$) to his qubit. If the measurement result of ν is one, he applies the Z gate ($= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$). Bob's qubit is now in the same state as the qubit Alice wanted to send. The circuit for this is shown in figure 2.9.

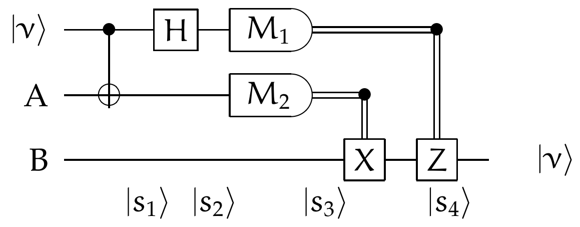


Figure 2.9: Quantum teleportation

For comparison , see figure C.1 on page 145 showing how this would be implemented in L-QPL. Additionally, a semantic proof of the correctness of the quantum teleportation protocol is discussed by Abramsky and Coecke in [AC04] and [Abr04].

[1]Notice that all other physical constraints are ignored in this algorithm. There is no concern about how one separates the qubits, transports the qubit or potential decoherence of the qubits.

Deutsch-Jozsa algorithm

The Deutsch-Jozsa algorithm describes a way of determining whether a function [2] f is *constant* (i.e. always 0 or 1) or *balanced* (i.e. produces an equal number of 0 or 1 results) based on applying it to one quantum bit. The function takes n bits as input and produces a single bit.

The function f is assumed to be expensive, therefore, it is desired to evaluate f as few times as possible before determining if f is balanced or constant. The worst case scenario when evaluating f classically is that determining the result requires $2^{n-1}+1$ invocations of the function. The best possible case is 2 invocations, which occurs when f is balanced and the first two inputs chosen produce different results.

The quantum circuit requires only one application of the function to $n+1$ qubits which have been suitably prepared to make the decision.

The algorithm relies on being able to construct an $n+1$ order unitary operator based upon f. In general, a unitary operator like this may be constructed by mapping the state $|a, b\rangle$ to $|a, b \oplus f(a)\rangle$ where $\oplus$ is the exclusive-or operator and a is n bit values. If we name this operator U_f, the circuit in figure 2.10 will solve the problem with just one application. See the appendix, appendix C.1.3 on page 146 for how this would be done in L-QPL.

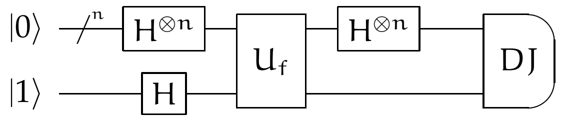


Figure 2.10: Circuit for the Deutsch-Jozsa algorithm

The idea of quantum parallelism is what makes this and many other quantum algorithms work. The initial state of the system is set to $|0^{\otimes n} \otimes 1\rangle$ after which the

[2]The required pre-condition for the Deutsch-Jozsa algorithm is that the function f is *either* balanced or constant and not some general function. The results are not well-defined if f does not fit into one of the two possible categories.

Hadamard gate is applied to all of the qubits. This places the input qubits into a superposition of all possible input values and the answer qubit is a superposition of 0 and 1. At this point, the unitary transformation U_f is applied to the qubits. Then the Hadamard transform is applied again to the input qubits.

To complete the algorithm, measure *all* the qubits. It can be shown that if f is constant, the input qubits will all measure to 0, while if it is balanced, at least one of those qubits will be 1.

Quantum Fourier transform

The circuits for the quantum Fourier transformation and its inverse are in figure 2.11 and figure 2.12 respectively. These transforms are used extensively in many quantum algorithms, including order finding in Shor's factoring algorithm.

The quantum Fourier transform is definable on an arbitrary number of qubits. This is typically presented by eliding the 3rd to the $n - 3$rd lines and interior processing. The L-QPL code for the quantum Fourier transform is in the appendix, figure C.2 on page 146. In this circuit, the parametrized transform R_n is the rotation transform, given by:

$$R_n = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{2^n}} \end{bmatrix}$$

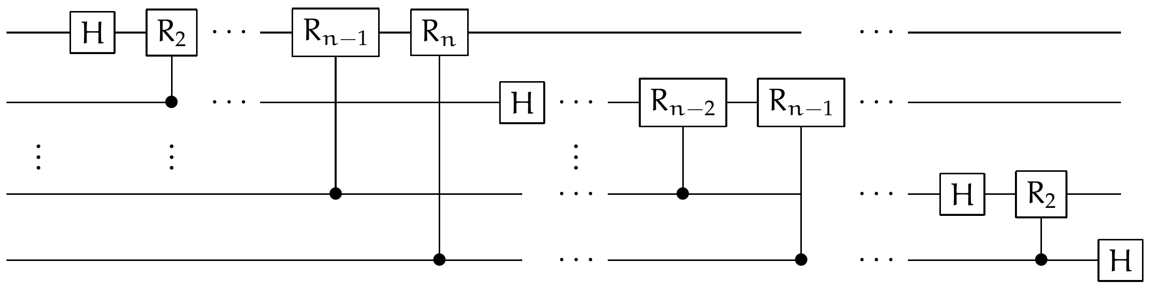


Figure 2.11: Circuit for the quantum Fourier transform

The inverse of a circuit is determined by traversing the circuit from right to left.

This process changes the original quantum Fourier circuit to its inverse as shown in figure 2.12. The L-QPL code for the inverse quantum Fourier transform is in the appendix, figure C.28 to figure C.29 on page 168.

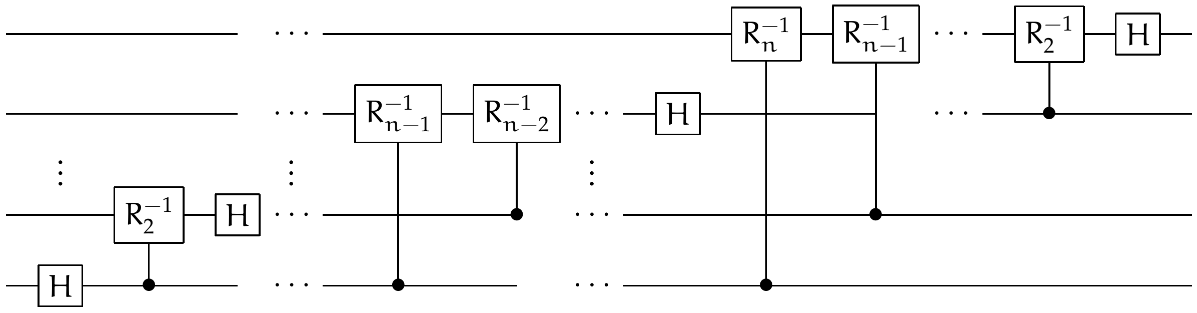


Figure 2.12: Circuit for the inverse quantum Fourier transform

2.4 Extensions to quantum circuits

To facilitate the transition to the programming language L-QPL, this section introduces three extensions to quantum circuits. The extensions are *renaming*, *wire bending and crossing*, and *scoped control*. Each extension adds expressive power to quantum circuits but does not change the semantic power. For each of the extensions, examples of how to re-write the extension in standard quantum circuit terminology will be provided.

2.4.1 Renaming

Quantum circuits currently allow renaming to be an implicit part of any gate. The circuit in figure 2.13 gives an operation to explicitly do this and its rewriting in standard circuit notation.

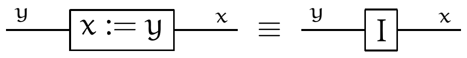


Figure 2.13: Renaming of a qubit and its equivalent diagram

2.4.2 Wire crossing

Crossing and bending of wires in a circuit diagram is added to allow a simpler presentation of algorithms. The circuit in figure 2.14 illustrates the concept of re-organizing and bending of wires.

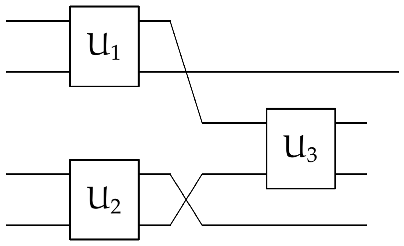


Figure 2.14: Bending

2.4.3 Scoped control

This extension allows us to group different operations in a circuit and show that all of them are controlled by a particular qubit. This is the same as attaching separate control wires to each of the gates in the grouped operations. Measurements are not affected by control. Figure 2.15 shows a scoped control box on the left which includes a measure. The right hand side of the same figure shows the circuit translated back to standard circuit notation, with the measure not being affected by the control.

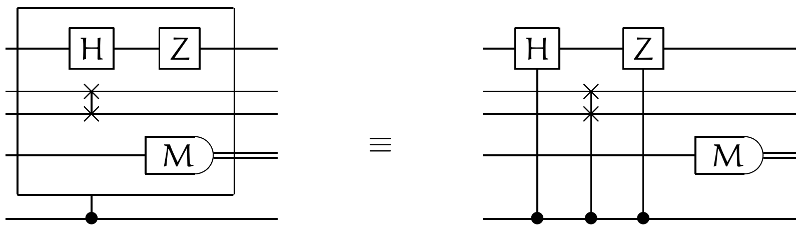


Figure 2.15: Scope of control

Scoping boxes correspond to procedures and blocks in L-QPL.

Naturally, both scoping and bending may be combined as in figure 2.16.

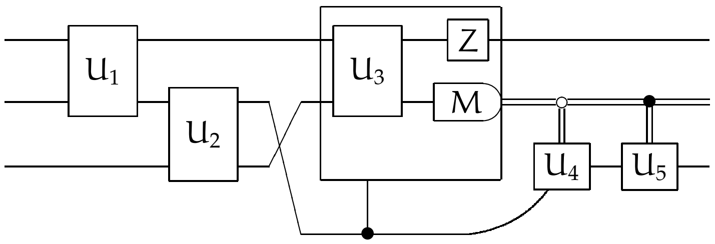


Figure 2.16: Extensions sample

However, note that exchanging wires is not the same as swap. Exchanging a pair of wires is not affected by control, but a swap is affected by control, as shown in figure 2.17.

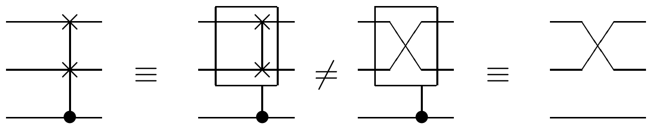


Figure 2.17: Swap in control vs. exchange in control

2.4.4 Circuit identities

Circuits allow the writing of the same algorithm in multiple ways. This sub-section will list some of the circuit identities that hold with the extended notation.

First, note that although a measure may appear inside a control box, it is not affected by the control, as in figure 2.18. Conversely, a measurement commutes with control of a circuit as in figure 2.18.

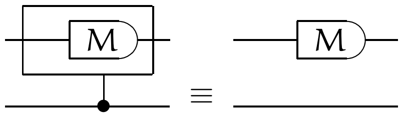


Figure 2.18: Measure is not affected by control

One of the notations introduced earlier was that of *0-control*. This type of control is the same as applying a Not transform before and after a *1-control*, as shown in figure 2.20 on the following page.

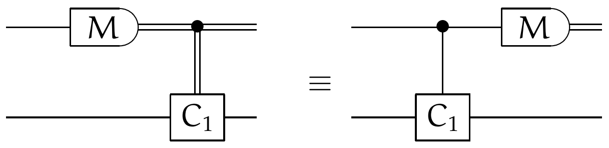


Figure 2.19: Control is not affected by measure

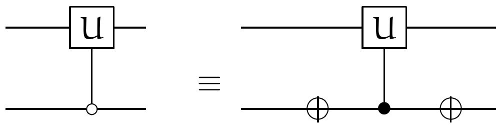


Figure 2.20: Zero control is syntactic sugar

Figure 2.21 shows that scoped control of multiple transforms is the same as controlling those transforms individually. Figure 2.22 similarly shows that scoped control

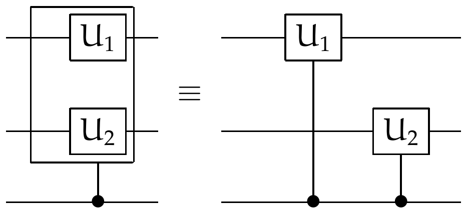


Figure 2.21: Scoped control is parallel control

of multiple transforms of the same qubit is the same as controlling those transforms serially.

Multiple control commutes with scoping as shown in figure 2.23 to figure 2.24 on the following page.

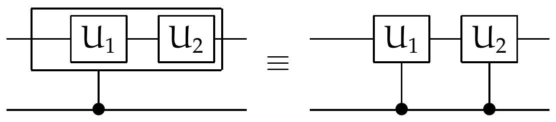


Figure 2.22: Scoped control is serial control

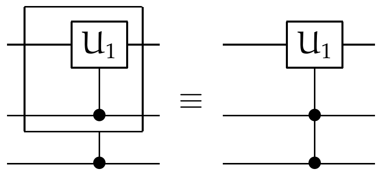


Figure 2.23: Multiple control

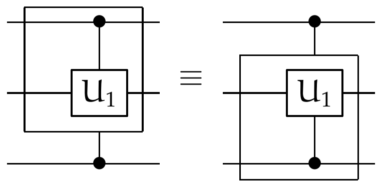


Figure 2.24: Control scopes commute

Chapter 3

Semantics

3.1 Basic L-QPL statements

Quantum circuits can be translated into basic L-QPL statements, which form a core fragment of the L-QPL language. Later sections in this chapter will provide the semantics for these statements and progressively build up to providing a semantics of L-QPL.

The language is introduced in a series of judgements which give the construction of valid L-QPL statements. Judgements have the following form:

$$\Gamma; \Gamma' \Vdash \mathcal{I}$$

where Γ is the context input to the statement $\mathcal{I}$ and Γ' is the context after it is completed. Notation of this form is becoming the standard way to communicate the meaning of language statements. For an thorough introduction to the subject, please see [Rey98] or [Win93].

Basic L-QPL has seven distinct statements: *identity, new, assign, discard, measure, control* and *unitary*. Statements may also be composed, creating a *block* of statements which may be used wherever a statement is required.

The quantum pseudocode conventions, as suggested by Knill in [Kni96] translate in reasonably obvious ways to these statements, with two exceptions. The main exception (functions) is covered later in this chapter. The other exception, reversability of quantum computations, is not handled by L-QPL directly. When the reverse of a particular computation is required, it would have to be directly coded.

The judgements for the creation of basic L-QPL statements are given in figure 3.1.

The *identity* statement is a do-nothing statement that does not change the stack or context. The statements *new* and *assign* share a similar syntax and create new variables in the output context. *Assign* will remove an existing variable, while *new* just creates the new variable. *Discard* removes a variable from the input context. The *measure* statement performs a destructive measure of a qubit, while applying different sets of dependent statements contingent on the result of the measure. *Control* modifies the execution of dependent statements, contingent upon the value of the attached control variable. The final statement, *unitary*, applies unitary transformations to a qubit.

$$\frac{}{\Gamma;\Gamma \Vdash \varepsilon}\ \text{identity}$$

$$\frac{i \in \{0,1\}}{\Gamma;(x{::}\mathtt{bit}),\Gamma \Vdash x = i}\ \text{new bit} \qquad \frac{k \in \{0,1\}}{\Gamma;(q{::}\mathtt{qubit}),\Gamma \Vdash x = |k\rangle}\ \text{new qubit}$$

$$\frac{}{(x{::}\tau),\Gamma;(y{::}\tau),\Gamma \Vdash x = y}\ \text{assign} \qquad \frac{}{(x{::}\tau),\Gamma;\Gamma \Vdash \text{disc } x}\ \text{discard}$$

$$\frac{\Gamma;\Gamma' \Vdash \mathfrak{I}_0 \quad \Gamma;\Gamma' \Vdash \mathfrak{I}_1}{(x{::}\mathtt{qubit}),\Gamma;\Gamma' \Vdash \text{meas } x\ \ |0\rangle => \mathfrak{I}_0\ \ |1\rangle => \mathfrak{I}_1}\ \text{measure}$$

$$\frac{\Gamma;\Gamma' \Vdash \mathfrak{I}}{(z{::}\tau),\Gamma;(z{::}\tau),\Gamma' \Vdash \mathfrak{I} <= z}\ \text{control} \qquad \frac{\Gamma;\Gamma' \Vdash \mathfrak{I}_1 \quad \Gamma';\Gamma'' \Vdash \mathfrak{I}_2}{\Gamma;\Gamma'' \Vdash (\mathfrak{I}_1;\mathfrak{I}_2)}\ \text{compose}$$

$$\frac{\Gamma_1;\Gamma_1' \Vdash \mathfrak{I}_1 \quad \Gamma_2;\Gamma_2' \Vdash \mathfrak{I}_2}{\Gamma_1\Gamma_2;\Gamma_1'\Gamma_2' \Vdash (\mathfrak{I}_1;;\mathfrak{I}_2)}\ \text{tensor compose}$$

$$\frac{}{(x{::}\mathtt{qubit}),\Gamma;(x{::}\mathtt{qubit}),\Gamma \Vdash U\ x}\ \text{unitary}$$

Figure 3.1: Judgements for creation of basic L-QPL statements.

3.1.1 Examples of basic L-QPL programs

The examples in this sub-section use the statements as introduced in figure 3.1 to give some example programs.

Program to swap two qubits. This program will swap two qubits by successive renaming. Both the input and output contexts have two qubits.

$q, v :: \mathtt{qubit}\ ;\ v, q :: \mathtt{qubit} \Vdash$

```
w = q;
q = v;
v = w;
```

Program to do a coinflip. The program has an empty input context and a single bit in its output context. At the end, the bit b will be 0 with probability .5 and 1 with an equal probability.

$\emptyset\ ;\ b :: \mathtt{bit} \Vdash$

```
q = |0>;
Had q;
meas q
  |0> => {b = 0}
  |1> => {b = 1}
```

Program to entangle two qubits. This program places two input qubits into an EPR state.

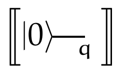
$q, r :: \mathtt{qubit}\ ;\ q, r :: \mathtt{qubit} \Vdash$

```
Had q;
Not r  ⇐  q;
```

3.2 Translation of quantum circuits to basic L-QPL

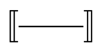
Quantum circuits are translatable to L-QPL statements. The semantic meaning of a circuit is denoted by enclosing the circuit between $[\![$ and $]\!]$.

When a new wire is added to a diagram (at the start of the diagram, this can be done for all wires) the meaning is an assignment of $|0\rangle$ to that variable.



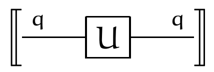
$$= q = |0\rangle \qquad (3.1)$$

An existing line with no operations on it translates to an identity statement:



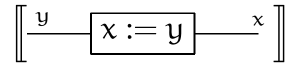
$$= \varepsilon. \qquad (3.2)$$

A unitary transform U translates to the obvious corresponding L-QPL statement:



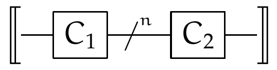
$$= U\ q. \qquad (3.3)$$

A renaming of a qubit wire translates to an assign statement:



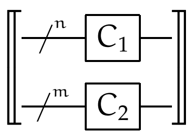
$$= x = y. \qquad (3.4)$$

A quantum circuit followed by another circuit is the composition of the meanings:



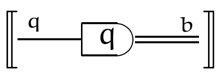
$$= [\![C_1]\!]\,;[\![C_2]\!]\,. \qquad (3.5)$$

A circuit that is above another is the tensor composition of the two circuits meanings:



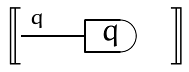
$$= [\![C_1]\!]\,;;[\![C_2]\!]\,. \qquad (3.6)$$

Measurement of a qubit and outputting a bit translates to the measure instruction:



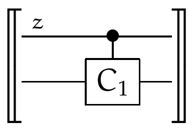
$$= \text{meas } q\ |0\rangle \Rightarrow \{b = 0\}\ |1\rangle \Rightarrow \{b = 1\}. \qquad (3.7)$$

A destructive measure, where the resulting bit is discarded, is translated as a discard:



$$= \text{disc } q\,. \qquad (3.8)$$

Control of a circuit translates to L-QPL control constructions. The controlling variable may be a qubit or bit:



$$= [\![C_1]\!] \Leftarrow z. \tag{3.9}$$

Equation (3.1) through equation (3.9) give a meaning for all the standard quantum circuit elements and their extensions described in section 2.4 on page 25. Certain constructs such as $0 - \texttt{control}$ have not been included as they are describable in terms of other circuit elements that have been assigned a semantics. See section 2.4 on page 25 for those identities.

3.3 Quantum stacks

This section will define a quantum stack and use quantum stacks to provide an operational semantics for L-QPL statements. The previous section has provided a translation of quantum circuits into these statements.

3.3.1 Definition of a quantum stack

In classical computing, a *stack* is an object together with operations for pushing new items onto the stack, retrieving items from the stack and examining whether the stack is empty or not. Refinements are often made in terms of adding linear addressing to the stack, multiple push-pull operations or defining specific data elements that are allowed on the stack.

A *quantum stack* is a somewhat more complex object, due to the properties of entanglement and superpositions. However, the basic idea of an object together with methods of adding and removing elements is retained. The initiating idea of the quantum stack was to find a stack-like way to represent the state of a quantum system and therefore the density matrix describing it.

The construction of a quantum stack is given by four judgements, shown in figure 3.2. These judgements show the quantum stack is a pair, $\Gamma \vdash S$. The Γ, called the *context*, is a list of names and their types. The S, called the *stack*, contains the actual stack nodes and traces. The stack will often be written as S^t where the t is an explicitly shown trace.

The judgement rules do not restrict quantum stacks to be Hermitian (i.e., correspond to a tuple of matrices, all of which are Hermitian) or to have a trace less than 1. However, when applied to a quantum stack with these properties, the statements used to interpret quantum circuits do not increase the trace and take Hermitian stacks to Hermitian stacks.

$$\frac{}{\vdash \emptyset^0} \qquad \frac{}{\vdash \alpha^{\sqrt{\alpha\overline{\alpha}}}}\ \text{scalar}$$

$$\frac{\Gamma \vdash S_{00}^{t_{00}}, S_{01}^{t_{01}}, S_{10}^{t_{10}}, S_{11}^{t_{11}}}{q{::}\mathrm{qubit}, \Gamma \vdash q\{ij \to S_{ij}\}^{t_{00}+t_{11}}}\ \text{qubit}$$

$$\frac{\Gamma \vdash S_0^{t_0}, S_1^{t_1}}{b{::}\mathrm{bit}, \Gamma \vdash b\{i \to S_i\}^{t_0+t_1}}\ \text{bit}$$

Figure 3.2: Judgements for a quantum stack

Description of the quantum stack

A quantum stack is a tree where each node is associated with a particular bit or qubit. The nodes associated with a qubit have four branches, corresponding to the density matrix representation of a qubit. The nodes associated with a bit have two branches, corresponding to the probabilistic representation of a bit. (i.e., bit $b = \alpha 0 + \beta 1$, where $\alpha + \beta \leqslant 1.0$). All nodes have a *trace*. Data values, which are the result of multiplying the probabilities of bits and density matrix entries for qubits, are stored at the leaves of the tree.

The trace of a quantum stack is a non-negative real number. It is defined recursively as shown in figure 3.3.

$$\mathrm{trace}(Q) = \begin{cases} \mathrm{trace}(Q_{00}) + \mathrm{trace}(Q_{11}) & \text{when } Q = q\left\{\begin{array}{ll} 00 \to Q_{00}; & 01 \to Q_{01} \\ 10 \to Q_{10}; & 11 \to Q_{11} \end{array}\right\} \\ \mathrm{trace}(B_0) + \mathrm{trace}(B_1) & \text{when } Q = b\{0 \to B_0; 1 \to B_1\} \\ \sqrt{v\bar{v}} & \text{when } Q = v \text{ (a scalar at the leaf)} \end{cases}$$

Figure 3.3: Definition of the trace of a quantum stack

Examples

The quantum stack of a single bit, b, which is 0 with a probability of .25 and 1 with a probability of .75, is represented as:

$$b\{0 \to .25; 1 \to .75\}^{1.0}.$$

A qubit q that was 0 and then subjected to the Hadamard transform followed by a Phase transformation is represented as:

$$q\{00 \to .5; 01 \to -.5i; 10 \to .5i; 11 \to .5\}^{1.0}.$$

A general qubit r having the density matrix $\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix}$ is represented as

$$r\{ij \to a_{ij}\}^{(a_{00}+a_{11})}$$

where the ij of the labelling correspond to the row and column of q's density matrix. The number at the upper right of the node representation is its trace.

Given a single bit b as above and a qubit q_1 which is 0, the representation is

$$b\{0 \to q_1\{00 \to .25\}^{.25}; 1 \to q_1\{00 \to .75\}^{.75}\}^{1.0}.$$

As shown in this example, values that are zero probabilities or zero entries of the matrix can be elided without loss of meaning.

3.3.2 Quantum stack equivalence

The equality judgements for quantum stacks are given in figure 3.4.

$$\frac{\Gamma \vdash S}{\Gamma \vdash S = \Gamma \vdash S}\text{ reflexivity} \qquad \frac{\Gamma \vdash S_2 = \Gamma \vdash S_1}{\Gamma \vdash S_1 = \Gamma \vdash S_2}\text{ symmetry}$$

$$\frac{\Gamma \vdash S_1 = \Gamma \vdash S_2 \quad \Gamma \vdash S_2 = \Gamma \vdash S_3}{\Gamma \vdash S_1 = \Gamma \vdash S_3}\text{ transitivity}$$

$$\frac{\Gamma, (t{::}\tau), \Gamma' \vdash S}{\Gamma, (t{::}\tau), \Gamma' \vdash S = (\text{rotate } t) \cdot \Gamma, (t{::}\tau), \Gamma' \vdash S}\text{ rotation}$$

Figure 3.4: Judgements for quantum stack equality

Rotation brings a desired node to the top of the stack and then reorganizes the tree so that the Huffman-like encoding of the leaves is invariant. Given a tree with four nodes (a, b, c, d), each of the leaves can be encoded by the indexes depending on the node names. e.g., $v_{i_a;i_b;i_c;i_d}$. Then, after rotating a tree, following a particular index path in the tree will take you to the same node. That is, if the tree order is now (d, b, c, a) with d, c qubits, a, b bits, then following the path $01 \to 1 \to 10 \to 0$ will take you to the leaf $v_{0;1;10;01}$, where the labelling is from the original a, b, c, d ordering. An example of this in the bit, qubit case is provided below.

Rotation example

Consider the following single bit, single qubit quantum stack,

$$c\left\{\begin{array}{l} 0 \to r\left\{\begin{array}{ll} 00 \to v_{0;00}; & 01 \to v_{0;01} \\ 10 \to v_{0;10}; & 11 \to v_{0;11} \end{array}\right\}^{v_{0;00}+v_{0;11}} \\ 1 \to r\left\{\begin{array}{ll} 00 \to v_{1;00}; & 01 \to v_{1;01} \\ 10 \to v_{1;10}; & 11 \to v_{1;11} \end{array}\right\}^{v_{1;00}+v_{1;11}} \end{array}\right\}^{\sum v},$$

where $\sum \nu = \nu_{0;00} + \nu_{0;11} + \nu_{1;00} + \nu_{1;11}$. If the qubit r is rotated to the top of the quantum stack, the resulting stack will be

$$r\left\{\begin{array}{ll} 00 \to c\{0 \to \nu_{0;00}; 1 \to \nu_{1;00}\}^{\nu'_{00}}; & 01 \to c\{0 \to \nu_{0;01}; 1 \to \nu_{1;01}\}^{\nu'_{01}} \\ 10 \to c\{0 \to \nu_{0;10}; 1 \to \nu_{1;10}\}^{\nu'_{10}}; & 11 \to c\{0 \to \nu_{0;11}; 1 \to \nu_{1;11}\}^{\nu'_{11}} \end{array}\right\}^{\sum \nu},$$

where $\sum \nu$ is as before and $\nu'_{ij} = \nu_{0;ij} + \nu_{1;ij}$.

Details of rotation on a quantum stack

Rotation is defined recursively on a quantum stack. Given a target name t and a quantum stack $\Gamma \vdash S$, rotation brings the node named t to the top and thus changes $\Gamma \vdash S$ to the quantum stack $(t : \tau), \Gamma' \vdash S'$. The quantum stack $\Gamma \vdash S$ is equivalent to $(t : \tau), \Gamma' \vdash S'$.

If a quantum stack has duplicate names, only the highest node with that name will be rotated up. Rotation's definition can be broken into cases depending on whether the target name is at the top, the second element or somewhere lower in the stack and whether the nodes being affected are bits or qubits.

When the target name is the top element of the stack, there is no change to the stack and rotation is just an identity transformation of the quantum stack.

In the case where the target name is the second element of the stack, there are four separate cases to consider: bit above a bit; bit above a qubit; qubit above a qubit and qubit above a bit.

bit above a bit:

$$(\text{rotate}_1\ b_2) \cdot b_1 \left\{\begin{array}{l} 0 \to b_2\{0 \to S_{0;0}; 1 \to S_{0;1}\} \\ 1 \to b_2\{0 \to S_{1;0}; 1 \to S_{1;1}\} \end{array}\right\}^{t} = b_2 \left\{\begin{array}{l} 0 \to b_1\{0 \to S_{0;0}; 1 \to S_{1;0}\} \\ 1 \to b_1\{0 \to S_{0;1}; 1 \to S_{1;1}\} \end{array}\right\}^{t}.$$

bit above a qubit:

$$(\text{rotate}_1\ q_2)\cdot b_1\left\{\begin{array}{l} 0\to q_2\left\{\begin{array}{ll}00\to S_{0;00}; & 01\to S_{0;01}\\ 10\to S_{0;10}; & 11\to S_{0;11}\end{array}\right\}\\ 1\to q_2\left\{\begin{array}{ll}00\to S_{1;00}; & 01\to S_{1;01}\\ 10\to S_{1;10}; & 11\to S_{1;11}\end{array}\right\}\end{array}\right\}^t$$

$$= q_2\left\{\begin{array}{ll}00\to b_1\{0\to S_{0;00};1\to S_{1;00}\}; & 01\to b_1\{0\to S_{0;01};1\to S_{1;01}\}\\ 10\to b_1\{0\to S_{0;10};1\to S_{1;10}\}; & 11\to b_1\{0\to S_{0;11};1\to S_{1;11}\}\end{array}\right\}^t.$$

qubit above a bit:

$$(\text{rotate}_1\ b_2)\cdot q_1\left\{\begin{array}{l}00\to b_2\{0\to S_{00;0};1\to S_{00;1}\}\\ 01\to b_2\{0\to S_{01;0};1\to S_{01;1}\}\\ 10\to b_2\{0\to S_{10;0};1\to S_{10;1}\}\\ 11\to b_2\{0\to S_{11;0};1\to S_{11;1}\}\end{array}\right\}^t$$

$$= b_2\left\{\begin{array}{l}0\to q_1\{00\to S_{00;0};01\to S_{01;0};10\to S_{10;0};11\to S_{11;0}\}\\ 1\to q_1\{00\to S_{00;1};01\to S_{01;1};10\to S_{10;1};11\to S_{11;1}\}\end{array}\right\}^t.$$

qubit above a qubit:

$$(\text{rotate}_1\ q_2)\cdot q_1\left\{\begin{array}{l} 00\to q_2\left\{\begin{array}{ll}00\to S_{00;00}; & 01\to S_{00;01}\\ 10\to S_{00;10}; & 11\to S_{00;11}\end{array}\right\}\\ 01\to q_2\left\{\begin{array}{ll}00\to S_{01;00}; & 01\to S_{01;01}\\ 10\to S_{01;10}; & 11\to S_{01;11}\end{array}\right\}\\ 10\to q_2\left\{\begin{array}{ll}00\to S_{10;00}; & 01\to S_{10;01}\\ 10\to S_{10;10}; & 11\to S_{10;11}\end{array}\right\}\\ 11\to q_2\left\{\begin{array}{ll}00\to S_{11;00}; & 01\to S_{11;01}\\ 10\to S_{11;10}; & 11\to S_{11;11}\end{array}\right\}\end{array}\right\}^t$$

$$= q_2\left\{\begin{array}{l} 00\to q_1\left\{\begin{array}{ll}00\to S_{00;00}; & 01\to S_{01;00}\\ 10\to S_{10;00}; & 11\to S_{11;00}\end{array}\right\}\\ 01\to q_1\left\{\begin{array}{ll}00\to S_{00;01}; & 01\to S_{01;01}\\ 10\to S_{10;01}; & 11\to S_{11;01}\end{array}\right\}\\ 10\to q_1\left\{\begin{array}{ll}00\to S_{00;10}; & 01\to S_{01;10}\\ 10\to S_{10;10}; & 11\to S_{11;10}\end{array}\right\}\\ 11\to q_1\left\{\begin{array}{ll}00\to S_{00;11}; & 01\to S_{01;11}\\ 10\to S_{10;11}; & 11\to S_{11;11}\end{array}\right\}\end{array}\right\}^t.$$

When the node to be rotated up is deeper in the quantum stack, we recurse down. Otherwise, we just apply rotate_1. That is:

$$(\text{rotate } n_3)\cdot n_1\{\ell_k\to n_2\{\ell'_m\to S\}\}$$

$$= (\text{rotate}_1\ n_3)\cdot n_1\{\ell_k\to (\text{rotate } n_3)\cdot n_2\{\ell'_m\to S\}\}$$

and

$$(\text{rotate } n_2) \cdot n_1\{\ell_k \to n_2\{\ell'_m \to S\}\} = (\text{rotate}_1\ n_2) \cdot n_1\{\ell_k \to n_2\{\ell'_m \to S\}\}.$$

In this way, the desired node "bubbles" to the top of the quantum stack.

3.3.3 Basic operations on a quantum stack

Quantum stacks may be added, tensored and multiplied by scalars. Multiplication of a quantum stack $\Gamma \vdash S$ by a scalar α recurses down the stack of S until the leaves are reached. Each leaf value is then multiplied by α. This will also have the effect of multiplying the trace of S by $\sqrt{\alpha\overline{\alpha}}$.

When adding two stacks, $\Gamma \vdash S$ and $\Gamma \vdash T$, the first step is to align the stacks. Aligning is done by recursively rotating nodes in the second stack to correspond with the same named nodes in the first stack. When the alignment is completed, the values at the corresponding leaves are added to produce a result stack. The trace of the resulting quantum stack will be the sum of the traces of S and T.

Tensoring of two quantum stacks $\Gamma_1 \vdash S_1$ and $\Gamma_2 \vdash S_2$ proceeds by first creating a new context Γ, which is the concatenation of Γ_1 and Γ_2. The new stack portion S is created by replacing each leaf value of S_1 with a copy of S_2 multiplied by the previous leaf value. The trace of the tensor will be the product of the traces of S_1 and S_2. The notation for this operation is

$$S = S_1 \otimes S_2.$$

The judgements for these stack operations are shown in figure 3.5 on the following page.

When recursing down stacks to do an addition or a scalar multiplication, any elided branches are treated as having the required structure with leaf values of 0.

$$\frac{\Gamma \vdash S^{t}}{\Gamma \vdash (\alpha S)^{t\sqrt{\alpha\overline{\alpha}}}}\ \text{scalar multiply}$$

$$\frac{\Gamma \vdash S_1^{t_1}, S_2^{t_2}}{\Gamma \vdash (S_1 + S_2)^{t_1+t_2}}\ \text{add}$$

$$\frac{\Gamma_1 \vdash S_1^{t_1} \quad \Gamma_2 \vdash S_2^{t_2}}{\Gamma_1\Gamma_2 \vdash (S_1 \otimes S_2)^{t_1\times t_2}}\ \text{tensor}$$

Figure 3.5: Judgements for basic stack operations

As an example, consider adding the following two quantum stacks:

$$S_1 = q\{00 \to b\{0 \to .1; 1 \to .1\}^{.2}; 11 \to b::\{1 \to .1\}^{.25}\}^{.45}$$

$$S_2 = q\{00 \to b\{0 \to .05; 1 \to .1\}^{.15}; 11 \to b\{0 \to .15; 1 \to .25\}^{.4}\}^{.55}.$$

In the first stage, recursively add the 00 branches of each, i.e.:

$$S_1[00] = b\{0 \to .1; 1 \to .1\}^{.2}$$

$$S_2[00] = b\{0 \to .05; 1 \to .1\}^{.15}.$$

When adding these two branches, each of the labels immediately point to leaves. As leaves are the base case for addition, this gives a result of:

$$S'[00] = b\{0 \to .15; 1 \to .2\}^{.35}.$$

Similarly, apply the addition process in the same manner to the 11 branches of S_1 and S_2:

$$S_1[11] = b\{1 \to .25\}^{.25}$$

$$S_2[11] = b\{0 \to .15; 1 \to .25\}^{.4}.$$

Note that in this case, the first sub-stack does not have a branch labelled by 0, due to elision of zero leaf values. The result of adding this is

$$S'[11] = \mathtt{b}\{0 \to .15; 1 \to .5\}^{.65},$$

which gives us the final result of:

$$S = \mathtt{q}\{00 \to \mathtt{b}\{0 \to .15; 1 \to .2\}^{.35}; 11 \to \mathtt{b}\{0 \to .15; 1 \to .5\}^{.65}\}^{1}.$$

The general cases for addition are given in figure 3.6. In the figure, the token $\langle \mathtt{empty} \rangle$ stands for any sub-branches that have been elided because all leaves below that node are 0. Sub-scripted upper case S's (S_i) represent quantum stacks, lower case sub-scripted v's (v_i) represent leaf values and sub-scripted curly l's (ℓ_i) are the labels of sub-stacks.

$$\begin{aligned}
S_1 + \langle \mathtt{empty} \rangle &= S_1\\
\langle \mathtt{empty} \rangle + S_2 &= S_2\\
v_1 + v_2 &= v_1 + v_2\\
v_1 + \langle \mathtt{empty} \rangle &= v_1\\
\langle \mathtt{empty} \rangle + v_2 &= v_2\\
\mathtt{nm}{::}\{\ell_i \to S_i\} + \mathtt{nm}{::}\{\ell_i \to S_i'\} &= \mathtt{nm}{::}\{\ell_i \to (S_i + S_i')\}\\
S_1 + S_2 &= S_1 + (\text{rotate } \mathtt{nm}_{S_1}) \cdot S_2.
\end{aligned}$$

Figure 3.6: Definition of addition of quantum stacks

3.4 Semantics of basic L-QPL statements

Section 3.1 introduced basic L-QPL statements, giving the judgements for their formation. This section introduces the concept of a statement modifier. Modifiers will be one

Notation	**Meaning**
$\widetilde{z}$	A list of names
$\Gamma;\Gamma' \Vdash \mathcal{I} \cdot \Gamma\Gamma'' \vdash S$	Application of the statements $\mathcal{I}$ to the quantum stack S.
$(\Gamma;\Gamma' \Vdash \mathcal{I} \Leftarrow \widetilde{z}) \cdot \Gamma \vdash S$	Application of the statements $\mathcal{I}$ to the quantum stack S, controlled by the bits and qubits in $\widetilde{z}$.
$(\Gamma;\Gamma' \Vdash \mathcal{I}) \cdot \Gamma \vdash S^t \rightsquigarrow \Gamma' \vdash S'^{t'}$	Application of the statements $\mathcal{I}$ with inputs Γ and output Γ' to the quantum stack S in context Γ results in the quantum stack S' in context Γ'.
$\Gamma \vdash \mathcal{I} \cdot S^t \rightsquigarrow \Gamma' \vdash S'^{t'}$	Alternate way to write the application of the statements $\mathcal{I}$ to the quantum stack S. Γ will contain the input variables of $\mathcal{I}$ and Γ' will contain the output variables.

Table 3.1: Notation used in judgements for operational semantics

of *IdOnly, Left*, or *Right*. The judgement for formation of statements with a modifier is:

$$\frac{\mathcal{M} \in \{\texttt{IdOnly, Left, Right}\} \quad \Gamma;\Gamma' \Vdash \mathcal{I}}{\Gamma;\Gamma' \Vdash \mathcal{M}\mathcal{I}}\ \text{modifier}$$

An operational semantics for basic L-QPL statements is given by the judgement diagrams in figure 3.7 on page 45 and figure 3.8 on page 46. Table 3.1 gives the additional notation used in the judgements.

The modifiers (`IdOnly, Left, Right`) come from examining the effect of a controlled transformation in a general setting. When a controlled transform and a general density matrix are multiplied:

$$\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & U^* \end{pmatrix} = \begin{pmatrix} A & BU^* \\ UC & UDU^* \end{pmatrix} \tag{3.10}$$

Equating the density matrix to the quantum stack notation, a controlled transformation is controlled by the qubit at the top of the quantum stack. It has the four sub-stacks A, B, C and D and the $\mathtt{controlled} - U$ transform is applied as in equation (3.10). The resulting quantum stack, with the same qubit at the top now has the four sub-stacks A, BU^*, UC and UDU^*. This leads to the obvious terminology of "`IdOnly` U" for the action on A, "`Right` U" for the action on B and "`Left` U" for the action on C. The action on D is a normal application of the transform U to D. As such, it does not need a modifier to describe the action.

Figure 3.7 on the following page presents the operational semantics of the the L-QPL statements *identity, new, discard, measure, compose* and *control*. Figure 3.8 on page 46 presents the operational semantics of *unitary* and the effects of statement modifiers.

3.4.1 Operational semantics commentary

Identity

This has no effect on the quantum stack or the context.

New bit and new qubit

These two statements transform the quantum stack S by adding a new node at the top of S. Both bit and qubit nodes are added with a single sub-branch which points to S. In the case of a bit the new sub-branch is labelled with 0, for a qubit the new sub-branch is labelled with 00. These labels mean the bit is 0 with 100% probability and the qubit is $1.0\,|0\rangle + 0.0\,|1\rangle$.

Delete bit and delete qubit

On the quantum stack S, deletion removes the top node x, merging the appropriate branches below x. Deletion of a bit merges both the 0 and 1 branches. Deletion of a qubit merges only the 00 and 11 branches.

$$\frac{}{\Gamma \vdash \varepsilon \cdot S \rightsquigarrow \Gamma \vdash S}\ \text{identity} \qquad \frac{}{\begin{array}{l}\Gamma \vdash (b = 0) \cdot S \rightsquigarrow \\ \qquad (b{::}\mathtt{bit}), \Gamma \vdash b\{0 \to S\}^{t}\end{array}}\ \text{new bit}$$

$$\frac{}{\Gamma \vdash (q = |0\rangle) \cdot S \rightsquigarrow (q{::}\mathtt{qubit}), \Gamma \vdash q\{00 \to S\}^{t}}\ \text{new qubit}$$

$$\frac{}{(b{::}\mathtt{bit}), \Gamma \vdash (\mathrm{disc}\ b) \cdot b\{0 \to S_0; 1 \to S_1\}^{t} \rightsquigarrow \Gamma \vdash (S_0 + S_1)^{t}}\ \text{delete bit}$$

$$\frac{}{(q{::}\mathtt{qubit}), \Gamma \vdash (\mathrm{disc}\ q) \cdot q\{ij \to S_{ij}\}^{t} \rightsquigarrow \Gamma \vdash (S_{00} + S_{11})^{t}}\ \text{delete qubit}$$

$$\frac{\Gamma \vdash I_0 \cdot S_{00}^{t_0} \rightsquigarrow \Gamma' \vdash S_0^{t_0'} \qquad \Gamma \vdash I_1 \cdot S_{11}^{t_1} \rightsquigarrow \Gamma' \vdash S_1^{t_1'}}{(q{::}\mathtt{qubit})\Gamma \vdash \left\{\begin{array}{l} \text{meas } q : \\ \ |0\rangle \Rightarrow I_0 \\ \ |1\rangle \Rightarrow I_1 \end{array}\right\} \cdot q\{ij \to S_{ij}\}^{t_0+t_1} \rightsquigarrow \Gamma' \vdash (S_0 + S_1)^{(t_0'+t_1')}}\ \text{measure}$$

$$\frac{\Gamma \vdash Op_1 \cdot S^{t} \rightsquigarrow \Gamma' \vdash S'^{t'} \qquad \Gamma' \vdash Op_2 \cdot S'^{t'} \rightsquigarrow \Gamma'' \vdash S''^{t''}}{\Gamma \vdash (Op_1; Op_2) \cdot S^{t} \rightsquigarrow \Gamma'' \vdash S''^{t''}}\ \text{compose}$$

$$\frac{\Gamma_1 \vdash Op_1 \cdot S_1^{t_1} \rightsquigarrow \Gamma_1' \vdash S_1'^{t_1'} \qquad \Gamma_2 \vdash Op_2 \cdot S_2{}^{t_2} \rightsquigarrow \Gamma_2' \vdash S_2'^{t_2'}}{\Gamma_1\Gamma_2 \vdash (Op_1;;Op_2) \cdot (S_1 \otimes S_2)^{t_1 \times t_2} \rightsquigarrow \Gamma_1'\Gamma_2' \vdash (S_1' \otimes S_2')^{t_1' \times t_2'}}\ \text{tensor}$$

$$\frac{\Gamma \vdash \mathcal{J} \Leftarrow \widetilde{z} \cdot S_1^{t} \rightsquigarrow \Gamma \vdash S'^{t'}_1}{\begin{array}{l}(z{::}\mathtt{bit}), \Gamma \vdash (\mathcal{J} \Leftarrow \widetilde{z}, z) \cdot z\left\{\begin{array}{l} 0 \to S_0^{t_0} \\ 1 \to S_1^{t} \end{array}\right\} \rightsquigarrow \\ \qquad\qquad (z{::}\mathtt{bit}), \Gamma \vdash z\left\{\begin{array}{l} 0 \to S_0 \\ 1 \to S_1' \end{array}\right\}^{t_0+t'}\end{array}}\ \text{control bit}$$

$$\frac{\begin{array}{c}\Gamma \vdash ((\mathtt{IdOnly}\ \mathcal{J}) \Leftarrow \widetilde{z}) \cdot S_{00} \rightsquigarrow \Gamma \vdash S_{00}' \\ \Gamma \vdash ((\mathtt{Left}\ \mathcal{J}) \Leftarrow \widetilde{z}) \cdot S_{10} \rightsquigarrow \Gamma \vdash S_{10}' \\ \Gamma \vdash ((\mathtt{Right}\ \mathcal{J}) \Leftarrow \widetilde{z}) \cdot S_{01} \rightsquigarrow \Gamma \vdash S_{01}' \\ \Gamma \vdash (\mathcal{J} \Leftarrow \widetilde{z}) \cdot S_{11}^{t_{11}} \rightsquigarrow \Gamma \vdash S'^{t_{11}'}_{11}\end{array}}{\begin{array}{l}(z{::}\mathtt{qubit}), \Gamma \vdash (\mathcal{J} \Leftarrow \widetilde{z}, z) \cdot z\{ij \to S_{ij}\} \rightsquigarrow \\ \qquad\qquad (z{::}\mathtt{qubit}), \Gamma \vdash z\left\{\begin{array}{ll} 00 \to S_{00}'; & 01 \to S_{01}' \\ 10 \to S_{10}'; & 11 \to S_{11}' \end{array}\right\}^{t_{00}+t_{11}'}\end{array}}\ \text{control qubit}$$

Figure 3.7: Operational semantics of basic L-QPL statements

$$\frac{\mathcal{U}\begin{pmatrix}S_{00} & S_{01}\\ S_{10} & S_{11}\end{pmatrix}\mathcal{U}^{*} = \begin{pmatrix}S'_{00} & S'_{01}\\ S'_{10} & S'_{11}\end{pmatrix}}{\begin{array}{l}(q{::}\mathtt{qubit}),\Gamma \vdash \mathcal{U}\ q \cdot q\{ij \to S_{ij}\}^{t} \rightsquigarrow \\ \qquad (q{::}\mathtt{qubit}),\Gamma \vdash q \left\{\begin{array}{ll} 00 \to S'_{00}; & 01 \to S'_{01} \\ 10 \to S'_{10}; & 11 \to S'_{11}\end{array}\right\}^{t}\end{array}}\ \mathcal{U}\text{ a transform}$$

$$\frac{}{\begin{array}{l}(q{::}\mathtt{qubit}),\Gamma \vdash \mathtt{IdOnly}\ \mathcal{U}\ q \cdot q\{ij \to S_{ij}\}^{t} \rightsquigarrow \\ \qquad (q{::}\mathtt{qubit}),\Gamma \vdash q\{ij \to S_{ij}\}^{t}\end{array}}\ \begin{array}{l}\mathtt{IdOnly}\ \mathcal{U},\\ \mathcal{U}\text{ a transform}\end{array}$$

$$\frac{\mathcal{U}\begin{pmatrix}S_{00} & S_{01}\\ S_{10} & S_{11}\end{pmatrix} = \begin{pmatrix}S'_{00} & S'_{01}\\ S'_{10} & S'_{11}\end{pmatrix}}{\begin{array}{l}(q{::}\mathtt{qubit}),\Gamma \vdash \mathtt{Left}\ \mathcal{U}\ q \cdot q\{ij \to S_{ij}\}^{t} \rightsquigarrow \\ \qquad (q{::}\mathtt{qubit}),\Gamma \vdash q \left\{\begin{array}{ll} 00 \to S'_{00}; & 01 \to S'_{01} \\ 10 \to S'_{10}; & 11 \to S'_{11}\end{array}\right\}^{t}\end{array}}\ \begin{array}{l}\mathtt{Left}\ \mathcal{U},\\ \mathcal{U}\text{ a transform}\end{array}$$

$$\frac{\begin{pmatrix}S_{00} & S_{01}\\ S_{10} & S_{11}\end{pmatrix}\mathcal{U}^{*} = \begin{pmatrix}S'_{00} & S'_{01}\\ S'_{10} & S'_{11}\end{pmatrix}}{\begin{array}{l}(q{::}\mathtt{qubit}),\Gamma \vdash \mathtt{Right}\ \mathcal{U}\ q \cdot q\{ij \to S_{ij}\}^{t} \rightsquigarrow \\ \qquad (q{::}\mathtt{qubit}),\Gamma \vdash q \left\{\begin{array}{ll} 00 \to S'_{00}; & 01 \to S'_{01} \\ 10 \to S'_{10}; & 11 \to S'_{11}\end{array}\right\}^{t}\end{array}}\ \begin{array}{l}\mathtt{Right}\ \mathcal{U},\\ \mathcal{U}\text{ a transform}\end{array}$$

$$\frac{\Gamma \vdash \mathcal{I} \cdot S \rightsquigarrow \Gamma' \vdash S'}{\Gamma \vdash (\mathcal{M}\mathcal{I}) \cdot S \rightsquigarrow \Gamma' \vdash S'}\ \mathcal{M}\mathcal{I}, (\mathcal{I}\text{ not a transform})$$

$$\frac{\Gamma \vdash (\mathcal{M}_1 \cdots \mathcal{M}_n)\mathcal{I} \cdot S \rightsquigarrow \Gamma' \vdash S' \quad \mathcal{M}_0 \equiv \mathcal{M}_1}{\Gamma \vdash (\mathcal{M}_0\, \mathcal{M}_1 \cdots \mathcal{M}_n)\mathcal{I} \cdot S \rightsquigarrow \Gamma' \vdash S'}\ \text{modifier reduction}$$

$$\frac{\mathcal{M}_0 \neq \mathcal{M}_1 \quad \Gamma \vdash (\text{IdOnly}\ \mathcal{I}) \cdot S \rightsquigarrow \Gamma' \vdash S'}{\Gamma \vdash ((\mathcal{M}_0\, \mathcal{M}_1 \cdots \mathcal{M}_n)\mathcal{I}) \cdot S \rightsquigarrow \Gamma' \vdash S'}\ \text{modifier elimination}$$

Figure 3.8: Operational semantics of basic L-QPL statements, continued

Measure

This statement applies separate groups of statements to the 00 and 11 sub-stacks of a qubit node, $\mathtt{q}$. After applying those statements, the node $\mathtt{q}$ is discarded.

Compose

Composition of the operations on a quantum circuit are done in left to right order of their application.

Control by bit

When the statement $\mathcal{I}_b$ is controlled by the bit z, the effect is to apply $\mathcal{I}_b$ only to the 1 sub-branch of z.

Control by qubit

If the statement $\mathcal{I}_q$ is controlled by the qubit w, different operations are applied to each of w's sub-branches. The modifiers `IdOnly`, `Left` and `Right` are used: (`IdOnly` $\mathcal{I}_q$) is applied to the the 00 sub-branch of w; (`Left` $\mathcal{I}_q$) to the 10 sub-branch; (`Right` $\mathcal{I}_q$) to the 01 sub-branch and the unmodified statement, $\mathcal{I}_q$, is applied to the 11 sub-branch.

Unitary transforms

Given the unitary transform $\mathtt{U}$ and a quantum stack $\mathtt{S}$ having the qubit $\mathtt{q}$ at the top, transforming $\mathtt{q}$ by $\mathtt{U}$ is defined in terms of matrix multiplication. Note that while figure 3.8 only gives the operational semantics for a single qubit transform, the same process is applied for higher degree transforms. A quantum stack with two qubits at its top may be converted to the 4×4 density matrix of those two qubits in the obvious way. A similar process may be used to extend the operation to more qubits.

Modifiers of unitary transforms

With U a unitary transform, the three rules for `IdOnly` U, `Left` U and `Right` U are given in terms of the appropriate matrix multiplications. As above, these may be extended to multiple qubit transformation in the obvious way.

Modifiers of statements other than unitary transforms

Modifiers have no effect on any statement of L-QPL except for unitary transformations.

Combinations of modifiers: reduction and elimination

When controlling by multiple qubits, it is possible to have more than one modifier applied to a statement. The modifier reduction rule states that applying the same modifier twice in a row has the same effect as applying it once.

Modifier elimination, however, reduces the list of modifiers to the single modifier `IdOnly` whenever two different modifiers follow one another in the modifier list.

3.5 Example — putting it all together on teleport

Circuit diagrams are interpreted in the algebra of quantum stacks. The preceding sections have given a translation of quantum circuits to L-QPL programs and an operational semantics of L-QPL programs. From this, we have an operational semantics of quantum circuits.

3.5.1 Translation of teleport to basic L-QPL

The quantum circuit for teleportation makes a variety of assumptions, including that the two starting qubits are placed in an EPR state and that the target qubit is in some unknown state. Figure 3.9 on the following page shows the standard teleport circuit assuming that ν is in some unknown state, but that the qubits *alice* and *bob* still need to be placed into an EPR state.

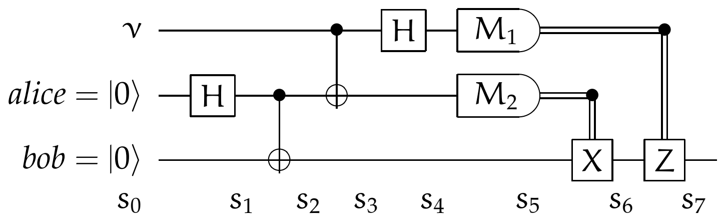


Figure 3.9: Preparation and quantum teleportation

The translation of this circuit to a basic L-QPL program is straightforward and is given in figure 3.10 on the next page.

3.5.2 Unfolding the operational semantics on teleport

This section will step through the program, using the operational semantics to show the changes to the quantum stack at each of the labelled stages. Input to the algorithm is assumed to be a single qubit, ν, which is in an unknown state, $\alpha\,|0\rangle + \beta\,|1\rangle$. Therefore, the starting stack is

$$\nu\left\{\begin{array}{ll} 00 \to \alpha\overline{\alpha}; & 01 \to \alpha\overline{\beta} \\ 10 \to \beta\overline{\alpha}; & 11 \to \beta\overline{\beta} \end{array}\right\}.$$

When the instructions $\mathcal{I}_{s_0}$ are applied, the stack will change to

$$bob\{00 \to alice\{00 \to \nu\left\{\begin{array}{ll} 00 \to \alpha\overline{\alpha}; & 01 \to \alpha\overline{\beta} \\ 10 \to \beta\overline{\alpha}; & 11 \to \beta\overline{\beta} \end{array}\right\}\}\}.$$

$$
\begin{aligned}
&\mathit{alice} = |0\rangle\ ; \\
&\mathit{bob} = |0\rangle\ ; && (\mathcal{I}_{s_0}) \\
&\mathsf{Had}\ \mathit{alice}; && (\mathcal{I}_{s_1}) \\
&(\mathsf{Not}\ \mathit{bob}) \Leftarrow \mathit{alice}; && (\mathcal{I}_{s_2}) \\
&\mathsf{Not}\ \mathit{alice} \Leftarrow \nu; && (\mathcal{I}_{s_3}) \\
&\mathsf{Had}\ \nu; && (\mathcal{I}_{s_4}) \\
&\quad \text{meas}\ \nu : \\
&\qquad |0\rangle \Rightarrow \nu_{\mathfrak{b}} = 0\ ; \\
&\qquad |1\rangle \Rightarrow \nu_{\mathfrak{b}} = 1 \\
&\quad \text{meas}\ \mathit{alice} : \\
&\qquad |0\rangle \Rightarrow \mathit{alice}_{\mathfrak{b}} = 0\ ; && (\mathcal{I}_{s_5}) \\
&\qquad |1\rangle \Rightarrow \mathit{alice}_{\mathfrak{b}} = 1 \\
&\mathsf{Not}\ \mathit{bob} \Leftarrow \mathit{alice}_{\mathfrak{b}}; \\
&\text{disc}\ \mathit{alice}_{\mathfrak{b}}; && (\mathcal{I}_{s_6}) \\
&\mathsf{Z}\ \mathit{bob} \Leftarrow \nu_{\mathfrak{b}}; \\
&\text{disc}\ \nu_{\mathfrak{b}} && (\mathcal{I}_{s_7})
\end{aligned}
$$

Figure 3.10: Basic L-QPL program for teleport

The next instruction, $\mathcal{I}_{s_1}$, applies the Hadamard transform to *alice*. *alice* is first rotated to the top and then the transformation is applied. The stack will then be in the state

$$alice\left\{\begin{array}{l}
00 \to bob\{00 \to \nu\left\{\begin{array}{ll} 00 \to .5 \times \alpha\overline{\alpha}; & 01 \to .5 \times \alpha\overline{\beta} \\ 10 \to .5 \times \beta\overline{\alpha}; & 11 \to .5 \times \beta\overline{\beta} \end{array}\right\}\} \\
01 \to bob\{00 \to \nu\left\{\begin{array}{ll} 00 \to .5 \times \alpha\overline{\alpha}; & 01 \to .5 \times \alpha\overline{\beta} \\ 10 \to .5 \times \beta\overline{\alpha}; & 11 \to .5 \times \beta\overline{\beta} \end{array}\right\}\} \\
10 \to bob\{00 \to \nu\left\{\begin{array}{ll} 00 \to .5 \times \alpha\overline{\alpha}; & 01 \to .5 \times \alpha\overline{\beta} \\ 10 \to .5 \times \beta\overline{\alpha}; & 11 \to .5 \times \beta\overline{\beta} \end{array}\right\}\} \\
11 \to bob\{00 \to \nu\left\{\begin{array}{ll} 00 \to .5 \times \alpha\overline{\alpha}; & 01 \to .5 \times \alpha\overline{\beta} \\ 10 \to .5 \times \beta\overline{\alpha}; & 11 \to .5 \times \beta\overline{\beta} \end{array}\right\}\}
\end{array}\right\}.$$

The preparation of EPR state is completed by doing a controlled-Not between *alice* and *bob*. This results in

$$alice\left\{\begin{array}{l}
00 \to bob\{00 \to \nu\left\{\begin{array}{ll} 00 \to .5 \times \alpha\overline{\alpha}; & 01 \to .5 \times \alpha\overline{\beta} \\ 10 \to .5 \times \beta\overline{\alpha}; & 11 \to .5 \times \beta\overline{\beta} \end{array}\right\}\} \\
01 \to bob\{10 \to \nu\left\{\begin{array}{ll} 00 \to .5 \times \alpha\overline{\alpha}; & 01 \to .5 \times \alpha\overline{\beta} \\ 10 \to .5 \times \beta\overline{\alpha}; & 11 \to .5 \times \beta\overline{\beta} \end{array}\right\}\} \\
10 \to bob\{01 \to \nu\left\{\begin{array}{ll} 00 \to .5 \times \alpha\overline{\alpha}; & 01 \to .5 \times \alpha\overline{\beta} \\ 10 \to .5 \times \beta\overline{\alpha}; & 11 \to .5 \times \beta\overline{\beta} \end{array}\right\}\} \\
11 \to bob\{11 \to \nu\left\{\begin{array}{ll} 00 \to .5 \times \alpha\overline{\alpha}; & 01 \to .5 \times \alpha\overline{\beta} \\ 10 \to .5 \times \beta\overline{\alpha}; & 11 \to .5 \times \beta\overline{\beta} \end{array}\right\}\}
\end{array}\right\}.$$

This is followed by the application of the controlled-Not to ν and *alice*, giving

$$\nu\left\{\begin{array}{l}
00 \to alice\left\{\begin{array}{ll} 00 \to bob\{00 \to .5\times\alpha\overline{\alpha}\}; & 01 \to bob\{10 \to .5\times\alpha\overline{\alpha}\} \\ 10 \to bob\{01 \to .5\times\alpha\overline{\alpha}\}; & 11 \to bob\{11 \to .5\times\alpha\overline{\alpha}\}\end{array}\right\} \\
01 \to alice\left\{\begin{array}{ll} 00 \to bob\{01 \to .5\times\alpha\overline{\beta}\}; & 01 \to bob\{11 \to .5\times\alpha\overline{\beta}\} \\ 10 \to bob\{00 \to .5\times\alpha\overline{\beta}\}; & 11 \to bob\{10 \to .5\times\alpha\overline{\beta}\}\end{array}\right\} \\
10 \to alice\left\{\begin{array}{ll} 00 \to bob\{00 \to .5\times\beta\overline{\alpha}\}; & 01 \to bob\{10 \to .5\times\beta\overline{\alpha}\} \\ 10 \to bob\{01 \to .5\times\beta\overline{\alpha}\}; & 11 \to bob\{11 \to .5\times\beta\overline{\alpha}\}\end{array}\right\} \\
11 \to alice\left\{\begin{array}{ll} 00 \to bob\{11 \to .5\times\beta\overline{\beta}\}; & 01 \to bob\{01 \to .5\times\beta\overline{\beta}\} \\ 10 \to bob\{10 \to .5\times\beta\overline{\beta}\}; & 11 \to bob\{00 \to .5\times\beta\overline{\beta}\}\end{array}\right\}
\end{array}\right\}.$$

The Hadamard transform is now applied to ν. Unfortunately, the notation used to this point is somewhat too verbose to show the complete resulting quantum stack. It begins with:

$$\nu\{00 \to alice\{00 \to bob\left\{\begin{array}{ll} 00 \to \frac{\alpha\overline{\alpha}}{4}; & 01 \to \frac{\alpha\overline{\beta}}{4} \\ 10 \to \frac{\beta\overline{\alpha}}{4}; & 11 \to \frac{\beta\overline{\beta}}{4}\end{array}\right\}\ldots\}\ldots\}$$
$$\vdots$$

In the next stage, *alice* is measured and a Not transform is applied to *bob* depending on the result of the measure. The bit $alice_{\mathtt{b}}$ is then discarded, resulting in

$$\nu \left\{ \begin{array}{l} 00 \to bob \left\{ \begin{array}{ll} 00 \to \alpha\overline{\alpha}/2; & 01 \to \alpha\overline{\beta}/2 \\ 10 \to \beta\overline{\alpha}/2; & 11 \to \beta\overline{\beta}/2 \end{array} \right\} \\ 01 \to bob \left\{ \begin{array}{ll} 00 \to \alpha\overline{\alpha}/2; & 01 \to -\alpha\overline{\beta}/2 \\ 10 \to \beta\overline{\alpha}/2; & 11 \to -\beta\overline{\beta}/2 \end{array} \right\} \\ 10 \to bob \left\{ \begin{array}{ll} 00 \to \alpha\overline{\alpha}/2; & 01 \to \alpha\overline{\beta}/2 \\ 10 \to -\beta\overline{\alpha}/2; & 11 \to -\beta\overline{\beta}/2 \end{array} \right\} \\ 11 \to bob \left\{ \begin{array}{ll} 00 \to \alpha\overline{\alpha}/2; & 01 \to -\alpha\overline{\beta}/2 \\ 10 \to -\beta\overline{\alpha}/2; & 11 \to \beta\overline{\beta}/2 \end{array} \right\} \end{array} \right\}.$$

Then, ν is measured and a RhoZ transform is applied when the resulting bit is 1. The bit $\nu_{\mathtt{b}}$ is then discarded. The resulting quantum stack is

$$bob \left\{ \begin{array}{ll} 00 \to \alpha\overline{\alpha}; & 01 \to \alpha\overline{\beta} \\ 10 \to \beta\overline{\alpha}; & 11 \to \beta\overline{\beta} \end{array} \right\},$$

which shows *bob* has been changed to the same state ν was in at the start of the program.

3.6 Semantics of datatypes

To this point, this chapter has only considered quantum stacks with two node types, bit and qubit. This section will add nodes for constructed datatypes and probabilistic classical data and statements that operate on these nodes.

Constructed datatypes will allow the addition of algebraic datatypes to our quantum stacks. This includes sum, product, and recursive data types. For example, types such as *List* (a recursive type), *Either* (a sum type) and *Pair* (a product type) are now definable.

3.6.1 Statements for constructed datatypes and classical data

Two new statements are required to implement datatypes on the quantum stack: *node construction* and a *case* statement. A third statement *discard data* is also added. *Discard data* is semantic sugar for doing a *case* without any dependent statements.

With the addition of classical data, arithmetic and logical *expressions* are added to L-QPL. The standard *new* and *discard* statements as well as a *use* statement are added to the language. The *if-else* statement is added to allow choices based on classical expressions.

The judgements for expressions are given in figure 3.11 and for the new statements in figure 3.12. Judgements for expressions use the notation

$$\Gamma_c|\Gamma \Vdash\!\dashv e :: \tau_C,$$

which means that given the context Γ_c of classical variables in the porch and the context Γ of quantum variables, e is a valid expression, with the classical type τ_C. Judgements for statements will also carry the classical context in the porch as above. Statements introduced to this point have not been affected by the classical context, nor have their operational semantics affected the classical context. The new format of judgements for statements will be:

$$\Gamma_c|\Gamma; \Gamma_c'|\Gamma' \Vdash \mathcal{I}.$$

For the operational semantics, the classical context must also now be carried in the porch. Additionally, the data in the classical context is not stored in the quantum stack, but in an adjacent standard stack. The full format of the semantics of a statement operating on the stacks will now be:

$$\Gamma_c|\Gamma \vdash \mathcal{I} \cdot (S, C) \rightsquigarrow \Gamma_c'|\Gamma' \vdash (S', C')$$

In many cases, the classical portions are not involved in the semantics and thus may be elided, resulting in our original syntax for the operational semantics.

$$\frac{n \in \mathbb{Z}}{\Gamma_c|\Gamma \Vdash n :: Int}\ \text{integers} \qquad \frac{n \in \{\text{true}, \text{false}\}}{\Gamma_c|\Gamma \Vdash b :: Boolean}\ \text{Booleans}$$

$$\frac{\Gamma_c|\Gamma \Vdash e_1 \quad \Gamma_c|\Gamma \Vdash e_2}{\Gamma_c|\Gamma \Vdash (e_1 \text{ op } e_2) :: \tau_{op}}\ \text{operations}$$

$$\frac{}{(n :: \tau_C), \Gamma_c|\Gamma \Vdash n :: \tau_C}\ \text{identifiers}$$

Figure 3.11: Judgements for expressions in L-QPL.

3.6.2 Operational semantics

Semantics for datatype statements

In the quantum stack, a datatype node of type τ has multiple branches, where each branch is labelled with a datatype constructor C of τ and the *bound nodes* for C. Constructors may require 0 or more bound nodes.

The *case* statement provides dependent statements to be executed for each of the branches of a datatype node d. Upon completion of the case, the node d is no longer available.

In discussing datatypes, the notation θ_z is used for creating a "fresh" variable. The notation $\tau(Cns)$ for the type of the constructor Cns.

The operational semantics for the datatype statements are given in figure 3.13. As *new data* and *discard data* have no effect on the classical context, the original syntax of the semantics statements is retained for these two statements. The *case* statement, however, does affect the classical context in that the classical context is reset at the beginning of execution of each set of dependent statements and at completion of the

$$\frac{}{(v_1 :: \tau_0, \ldots v_n :: \tau_n), \Gamma; (x :: \tau(C)), \Gamma \Vdash x = C(v_1, \ldots, v_n)}\ \text{new data}$$

$$\frac{\{\Gamma; \Gamma' \Vdash \mathcal{I}_i\}_i}{(nd :: \tau), \Gamma; \Gamma' \Vdash \left\{ \begin{array}{l} \text{case } nd : \\ \{\ C_i(v_{ij}) => \mathcal{I}_i\}_{C_i \in \tau(C_i)} \end{array} \right\}}\ \text{case}$$

$$\frac{}{(nd :: \tau), \Gamma; \Gamma \Vdash \text{disc } nd}\ \text{discard data}$$

$$\frac{\Gamma_c | \Gamma \Vdash\!\dashv e}{\Gamma_c | \Gamma; \Gamma_c | (x :: \tau_C), \Gamma \Vdash x = e}\ \text{new classical}$$

$$\frac{(n :: \tau_C), \Gamma_c | \Gamma; \Gamma_c | \Gamma' \Vdash \mathcal{I}}{\Gamma_c | (n :: \tau_C), \Gamma; \Gamma_c | \Gamma' \Vdash \{\text{use } n :\ \{\mathcal{I}\}\}}\ \text{use}$$

$$\frac{\{\Gamma_c | \Gamma \Vdash\!\dashv e_i{::}Boolean\}_{i=0,\ldots,n-1} \quad \{\Gamma_c | \Gamma; \Gamma_c^i | \Gamma' \Vdash \mathcal{I}_i\}_{i=0,\ldots,n}}{\Gamma_c | \Gamma; \Gamma_c | \Gamma' \Vdash \left\{ \begin{array}{l} \text{if } e_0 => \mathcal{I}_0 \\ \{\ e_i => \mathcal{I}_i\}_i \\ \text{else } => \mathcal{I}_n \end{array} \right\}}\ \text{if-else}$$

$$\frac{}{\Gamma_c | (n :: \tau_C), \Gamma; \Gamma_c' | \Gamma' \Vdash \text{disc } n}\ \text{discard classical}$$

Figure 3.12: Judgements for datatype and classical data statements.

statement. Therefore, the semantics for *case* will use the full syntax introduced in this section.

$$\frac{(x_i{::}\tau_i)_{i=1,\ldots m}, \Gamma \vdash ([\theta_{x_1}/x_1];\ldots;[\theta_{x_m};x_m]) \cdot S \rightsquigarrow (\theta_{x_i}{::}\tau_i)_{i=1,\ldots m}, \Gamma' \vdash S'}{\begin{array}{l}(x_i{::}\tau_i)_{i=1,\ldots m}, \Gamma \vdash (nd = Cns(x_1,\ldots,x_m)) \cdot S \rightsquigarrow \\ \quad (nd{::}\tau(Cns)), (\theta_{x_i}{::}\tau_i)_{i=1,\ldots m}, \Gamma' \vdash nd\{Cns(\theta_{x_1},\ldots,\theta_{x_m}) \to S'\}\end{array}} \text{ new data}$$

$$\frac{\{\Gamma \vdash (\{\text{disc } x_{ij}\}_j) \cdot S_i \rightsquigarrow \Gamma' \vdash S_i'\}_i}{(nd :: \tau), \Gamma \vdash \text{disc } nd \cdot nd\{C_i(x_{ij}) \to S_i\} \rightsquigarrow \Gamma' \vdash \sum S_i'} \text{ discard data}$$

$$\frac{\{\Gamma_c|\Gamma \vdash ([v_{ij}/x_{ij}])_j; \mathcal{I}_i \cdot (S_i, C) \rightsquigarrow \Gamma_c'|\Gamma' \vdash (S_i', C_i)\}_i}{\begin{array}{l}\Gamma_c|(nd :: \tau), \Gamma \vdash \left\{\begin{array}{l}\text{case } nd : \\ \{\ Cns_i(v_{ij}) => \mathcal{I}_i\}\end{array}\right\} \cdot (nd\{Cns_i(x_{ij}) \to S_i\}, C) \rightsquigarrow \\ \qquad\qquad \Gamma_c|\Gamma' \vdash (\sum S_i', C)\end{array}} \text{ case}$$

Figure 3.13: Operational semantics for datatype statements

Classical data

Adding probabilistic classical data led to adding a second data structure for the operational semantics. In order to perform standard arithmetic and logical (or other classically defined) operations on this data, a *classical stack* is paired with the quantum stack.

Classical data nodes extend the concept of probabilistic bits to other types. For example, a node c could hold an integer that had value 7 with probability .3, 5 with probability .2 and 37 with probability .5. A probabilistic node is restricted to items of all the same type, e.g., all integers, all floats.

The classical stack is used to operate on non-probabilistic classical data and will interact with the classical nodes via *classical construction* and *use*.

The *use* statement had a set of dependent statements, $\mathcal{I}$, which are executed for each sub-stack S_i of a a classical node c. Prior to executing $\mathcal{I}$, the classical value k_i

which labels the sub-branch S_i is moved onto the top of the classical stack. When the statements $\mathcal{I}$ have been executed for each of c's sub-stacks, c is removed and the resulting sub-branches are added together.

The classical stack has standard arithmetic and logic operations defined on it. The *if - else* statement is defined to execute various statements depending upon classical expressions.

The judgements for the classical note construction, discarding, use and interaction with the classical stack are shown in figure 3.14.

$$\frac{\Gamma_c|\Gamma \Vdash e}{\Gamma_c|\Gamma \vdash (n = e)\cdot(S,C) \rightsquigarrow \Gamma_c|(n{::}\tau_C),\Gamma \vdash (n\{e \to S\},C)}\ \text{new classical}$$

$$\frac{\{(n{::}\tau_C),\Gamma_c|\Gamma \vdash \mathcal{I}\cdot(S_i,k_i:C) \rightsquigarrow \Gamma_c'|\Gamma' \vdash (S_i',C_i')\}_i}{\Gamma_c|(n{::}\tau_C),\Gamma \vdash \{\text{use } nd:\ \{\mathcal{I}\}\}\cdot(nd\{k_i \to S_i\},C) \rightsquigarrow \Gamma_c|\Gamma' \vdash (\sum S_i',C)}\ \text{use}$$

$$\frac{}{\Gamma_c|(n{::}\tau_C),\Gamma \vdash \text{disc } n\cdot(nd\{cv_i \to S_i\},C) \rightsquigarrow \Gamma_c|\Gamma \vdash (\sum S_i,C)}\ \text{discard classical}$$

$$\frac{e_k = \mathtt{True}\quad \{e_j = \mathtt{False}\}_{j<k}\quad \Gamma_c|\Gamma \vdash \mathcal{I}_k\cdot(S,C) \rightsquigarrow \Gamma_c^k|\Gamma' \vdash (S',C_k)}{\Gamma_c|\Gamma \vdash \left\{\begin{array}{l}\text{if } e_0 => \mathcal{I}_0\\ \{\, e_i => \mathcal{I}_i\}_i\\ \text{else } => \mathcal{I}_n\end{array}\right\}\cdot(S,C) \rightsquigarrow \Gamma_c^k|\Gamma' \vdash (S',C_k)}\ \text{if-else, } e_k \text{ true}$$

$$\frac{\{e_j = \mathtt{False}\}_{j=0,\dots,n-1}\quad \Gamma_c|\Gamma \vdash \mathcal{I}_n\cdot(S,C) \rightsquigarrow \Gamma_c^n|\Gamma' \vdash (S',C_n)}{\Gamma_c|\Gamma \vdash \left\{\begin{array}{l}\text{if } e_0 => \mathcal{I}_0\\ \{\, e_i => \mathcal{I}_i\}_i\\ \text{else } => \mathcal{I}_n\end{array}\right\}\cdot(S,C) \rightsquigarrow \Gamma_c^n|\Gamma' \vdash (S',C_n)}\ \text{if-else, else}$$

$$\frac{v_i\, op_c\, v_j = v_k}{i :: \tau_1, j :: \tau_2, \Gamma_c|\Gamma \vdash (\text{capp } op_c)\cdot(S, v_i : v_j : C) \rightsquigarrow k :: \tau_3, \Gamma_c|\Gamma \vdash (S, v_k : C)}\ \text{binary op}$$

Figure 3.14: Operational semantics for classical data statements

An important point to note in the transitions for the `use` statement is that the classical stack is *reset* at the beginning of each execution and in the final result. This enforces a block-like scoping on the classical stack. This also applies to `measure` and `case`. The revised judgement for measure is given in figure 3.15.

$$\frac{\begin{array}{c}\Gamma_c|\Gamma \vdash I_0 \cdot (S_{00}, C) \rightsquigarrow \Gamma_c'|\Gamma' \vdash (S_0, C_0)\\ \Gamma_c|\Gamma \vdash I_1 \cdot (S_{11}, C) \rightsquigarrow \Gamma_c''|\Gamma' \vdash (S_1, C_1)\end{array}}{\Gamma_c|\Gamma \vdash \left\{\begin{array}{c}\text{meas } q :\\ |0\rangle => I_0\\ |1\rangle => I_1\end{array}\right\} \cdot (q\{ij \to S_{ij}\}, C) \rightsquigarrow \Gamma_c|\Gamma' \vdash ((S_0 + S_1), C)}\ \text{measure}'$$

Figure 3.15: Revised semantics for measure with classical stack.

3.7 Semantics of recursion

This section will extend L-QPL with statements for recursion and provide the operational semantics for them.

3.7.1 Statements for recursion

The only statements added for recursion are the *proc* and *call* statements. *Proc* is used to create a subroutine that is available for the *call* statement.

All *proc* statements are global in scope to a program, hence a subroutine may *call* itself or be called by other subroutines defined elsewhere in a program. The judgements for these two new statements are given in figure 3.16 on the next page. The judgements reuse the notation introduced earlier of $\widetilde{z_j}$ signifying a vector of elements.

3.7.2 Operational semantics for recursion

In order to provide an operational semantics for recursion, the quantum stacks will now be considered a *stream* (also known as an infinite list) of quantum stacks. The

$$\frac{}{\Gamma_c|\Gamma \Vdash \mathrm{prc}_x :: (\widetilde{c_i{:}\tau_{C_i}}|\widetilde{qv_j{:}\tau_j};\widetilde{r_k{:}\tau_k}) = \mathcal{I}}\ \text{proc}$$

$$\frac{\{\Gamma_c|(\widetilde{qv_j} :: \widetilde{\tau_j})\Gamma \Vdash\!\dashv e_i\}_i}{\Gamma_c|(\widetilde{qv_j} :: \widetilde{\tau_j})\Gamma \Vdash (\widetilde{r_k}) = \mathrm{prc}_c(\widetilde{e_i}|\widetilde{qv_j})}\ \text{call}$$

Figure 3.16: Judgements for formation of proc and call statements

Notation	**Meaning**
$[z'/z]\Gamma \vdash [z'/z]S$	Substituting z' for z in the context and stack.
$\Gamma_c|\Gamma \vdash \mathcal{I} \cdot (S, C) \overset{d}{\rightsquigarrow} \Gamma_c'\Gamma' \vdash (S', C')$	Application of the statements $\Gamma_c|\Gamma; \Gamma_c'\Gamma' \Vdash \mathcal{I}$ to to the quantum stack / classical stack pair (S, C) in classical context Γ_c and context Γ results in the quantum stack / classical stack pair (S', C') in classical context Γ_c' and context Γ' *at the* d^th^ *iteration*.

Table 3.2: Notation used in operational semantics once iteration is added

semantics as given previously in this chapter will stay the same, with the provision that all of these apply across the entire stream.

Additional notation is required for calling subroutines, re-namings (formal in and out parameters) and for applying a different transition dependent on the level in the stream. This additional notation is explained in table 3.2.

The operational semantics for recursion is given in figure 3.17. Recursion is done in the semantics by first *diverging* at the head of the infinite list and then calling a subroutine once at the second element, calling twice in the third and so forth. In this way, the program gives a closer and closer approximation of the actual results the further one looks down the stream of quantum stacks.

$$\frac{}{(x{::}T),\Gamma \vdash [x'/x]\cdot x\{v_i \longrightarrow S_i\}^t \rightsquigarrow (x'{::}T),\Gamma \vdash x'\{v_i \longrightarrow S_i\}^t}\ \text{rename}$$

$$\frac{\mathrm{prc}_a :: (\widetilde{c_i{:}\tau_{C\,i}} | \widetilde{z_j{:}\tau_j}; \widetilde{w_k{:}\tau_k}) = \mathcal{I}_a}{\Gamma_c|(\widetilde{z_j} :: \widetilde{\tau_j})\Gamma \vdash (\widetilde{w_k}) = \mathrm{prc}_a(\widetilde{c_i}|\widetilde{z_j}) \cdot (S, C) \overset{0}{\rightsquigarrow} \Gamma_c|\emptyset \vdash (\emptyset^0, C)}\ \text{base call}$$

$$\frac{\begin{array}{c}\Gamma_c|\Gamma \vdash \{[z'_j/z_j]\}_j; \mathcal{I}_a; \{[w_k/w'_k]\}_k \cdot (S, C) \overset{d}{\rightsquigarrow} \Gamma'_c|(\widetilde{w_k} :: \widetilde{\tau_k})\Gamma' \vdash (S', C') \\ \mathrm{prc}_a :: (\widetilde{c'_i{:}\tau_{C\,i}} | \widetilde{z'_j{:}\tau_j}; \widetilde{w'_k{:}\tau_k}) = \mathcal{I}_a \\ \{\Gamma_c|(\widetilde{z_j} :: \widetilde{\tau_j})\Gamma \mathrel{\Vdash\!\dashv} e_i{::}\tau_{C\,i}\}_i\end{array}}{\Gamma_c|(\widetilde{z_j} :: \widetilde{\tau_j})\Gamma \vdash (\widetilde{w_k}) = \mathrm{prc}_a(\widetilde{c_i}|\widetilde{z_j}) \cdot (S, C) \overset{d+1}{\rightsquigarrow} \Gamma_c|\Gamma' \vdash (S', C)}\ \text{call } d+1$$

Figure 3.17: Operational semantics for recursion

Chapter 4

An Informal Introduction to Linear QPL

4.1 Introduction to L-QPL

This chapter presents an overview of the linear quantum program language. Explanations of L-QPL programs, statements, and expressions are given. The explanations are done using expository presentation with many short examples to illustrate relevant points.

L-QPL is a language for experimenting with quantum algorithms. The language provides an expressive syntax for creating functions and defining and working with different datatypes. L-QPL has qubits as first class citizens of the language, together with quantum control. Classical operations and classical control are also available to work with classical data.

The language design started from QPL in [Sel04] and rapidly evolved to a point where a direct comparison is somewhat difficult. Major differences are the type system, the syntax and structure of functions and the choices of individual statements.

4.1.1 Functional versus imperative

L-QPL is a functional language that uses single assignment. It resembles QPL in this aspect rather than QML of [AG05]. Single assignment means that a variable always has a unique value until its use. (See sub-section 4.1.2 below.)

Like most functional languages, side effects are not allowed in functions, *other than those that occur due to quantum entanglement*. Side effects in imperative languages are those where a global variable is updated or values are read or written. An example of this in an imperative quantum language is a procedure in QCL from [Öme00]. L-QPL does not have the concept of a global variable. Currently, I/O is undefined. Side effects from quantum entanglement can occur when a qubit is passed as a parameter to a function and it is operated on in the function.

4.1.2 Linearity of L-QPL

The language L-QPL treats all quantum variables as *linear*. This means that any variable *may only be used once*. The primary reason for implementing this is the underlying aspect of linearity of quantum systems, as exemplified by the *no-duplication* rule which must be respected at all times. This allows us to provide compile-time checking that enforces this rule.

The compiler and language do provide ways to "ease the burden" of linear thinking. For example, function calls (sub-section 4.3.6 on page 77) provide a specialized syntax for variables which are both input and output to a function. The classical use statements (sub-section 4.3.5 on page 75) place values on to the classical stack where the values may be used multiple times.

```
1 #Import Prelude.qpl
2
3 len::(listIn:List(a) ; length:Int) =
4 {   case listIn of
5       Nil => {length = 0}
6       Cons (_, tail) =>
7         { tlLen := len(tail);
8           length = 1 + tlLen }
9 }
```

Figure 4.1: L-QPL code to return the length of the list

An example illustrating linearity is given in figure 4.1. In line 3, the function len is defined as taking one argument of type List(a) and returning a variable of type **Int**. The input only argument, listIn, *must be destroyed in the function*. When the case statement refers to listIn, the argument is destroyed, fulfilling the requirement of the function to do so.

4.2 L-QPL programs

L-QPL programs consist of combinations of functions and data definitions, with one special function named **main**. The functions and data definitions are *simultaneously declared* and so may be given in any order. The program will start executing at the **main** function.

A physical program will typically consist of one or more source files with the suffix `.qpl`. Each source file may contain functions and data definitions. It may also *import* the contents of other source files. The name of a source file is not significant in L-QPL. Common practice is to have one significant function per source file and then to import all these files into the source file containing the **main** function.

The above structure was chosen thinking of Haskell [Pey03] and C [KR88]. Global definitions of functions and types is borrowed from Haskell, while the **main** start point and import feature is a combination of Haskell and C.

4.2.1 Data definitions

L-QPL provides the facilities to define datatypes with a syntax reminiscent of Haskell [Pey03].

Natively, the language provides **Int**, **Qubit** and **Bool** types. **Bool** is the standard Boolean type with values true and false. **Int** is a standard 32-bit integer. **Qubit** is a single qubit.

In L-QPL both native types and other constructed datatypes may be used in the definition of constructed datatypes. These constructed datatypes may involve sums, products, singleton types and parametrization of the constructed type. For example, a type that is the sum of the integers and the Booleans can be declared as follows:

```
qdata Eitherib = {Left(Int) | Right(Bool)}
```

The above example illustrates the basic syntax of the data declaration.

Syntax of datatype declarations. Each datatype declaration must begin with the keyword **qdata**. This is followed by *the type name*, which must be an identifier starting with an uppercase letter. The type name may also be followed by any number of *type variables*. This is then followed by an equals sign and completed by a list of *constructors*. The list of constructors must be surrounded by braces and each constructor must be separated from the others by a vertical bar. Each constructor is followed by an optional parenthesized list of *simple types*. Each simple type is either a built-in type, (one of **Int**, **Bool**, **Qubit**), a type variable that was used in the type declaration, or another declared type, surrounded by parenthesis. L-QPL allows recursive references to the type currently being declared. All constructors must begin with an upper case letter. Constructors and types are in different namespaces, so it is legal to have the same name for both. For example:

```
qdata Record a b = {Record(Int, a, b)}
```

In the above type definition, the first Record is the type, while the second is the constructor. The triplet (**Int**,a,b) is the product of the type **Int** and the type variables a and b. Since constructors may reference their own type and other declared types, recursive data types such as lists and various types of trees may be created:

```
qdata List a   = {Nil     | Cons(a, List(a))}
qdata Tree a   = {Leaf(a) | Br(Tree(a), Tree(a))}
qdata STree a  = {Tip     | Fork(STree(a), a, STree(a))}
qdata Colour   = {Red      | Black}
qdata RBSet a  = {Empty    |
                  RBTip(Colour, RBSet(a), a, RBSet(a))}
qdata RTree a  = {Rnode(a, List(a))}
qdata Rose a   = {Rose(List(a, Rose(a)))}
```

4.2.2 Function definitions

Function definitions may appear in any order within a L-QPL source file.

Syntax of function definitions. The first element of a function definition is *the name*, an identifier starting with a lower case letter. This is always followed by a double semi-colon and *a signature*, which details the type and characteristics of input and output arguments. The final component of the function definition is *a body*, which is a block of L-QPL statements. Details of statements are given in section 4.3 on page 69.

The structure of function definitions is unique to L-QPL, although broadly based on one of the acceptable syntaxes for C function definitions as in [KR88]. The major difference occurs in the signature which, as will be seen below, allows for both classical and quantum input arguments and specifies the quantum outputs of a function. Another difference is that L-QPL defines all functions globally, hence there is no requirement for declarations separate from the definitions.

Let us examine two examples of functions. The first, in figure 4.2 on the next page is a fairly standard function to determine the greatest common divisor of two integers.

The first line has the name of the function, gcd, followed by the signature. The name and the signature are separated by a double colon. The signature of the function is (a:**Int**, b:**Int** | ; ans:**Int**). This signature tells the compiler that gcd expects two input arguments, each of type **Int** and that they are *classical*. The compiler deduces this from

```
gcd::(a:Int, b:Int | ; ans: Int) =
{ if b == 0 => {ans = a}
     a == 0 => {ans = b}
     a ⩾ b => {ans = gcd(b, a mod b)}
  else      => {ans = gcd(a, b mod a)}
}
```

Figure 4.2: L-QPL function to compute the GCD

the fact that they both appear before the '|'. In this case there are no quantum input arguments as there are no parameters between the '|' and the ';'. The last parameter tells us that this function returns one quantum item of type **Int**. Returned items are always quantum data.

The signature specifies variable names for the parameters used in the body. All input parameters are available as variable names in expression and statements. Output parameters are available to be assigned and, indeed, must be assigned by the end of the function.

The next example, in figure 4.3 on the following page highlights the linearity of variables in L-QPL. This function is used to create a list of qubits corresponding to the bit representation of an input integer.

At line 1, the program uses the import command. #Import must have a file name directly after it. This command directs the compiler to stop reading from the current file and to read code in the imported file until the end of that file, after which it continues with the current file. The compiler will not reread the same file in a single compilation, and it will import from any file.

For example, consider a case with three source files, A, B and C. Suppose file A has import commands for both B and C, with B being imported first. Further suppose that file B imports C. The compiler will start reading A, suspend at the first import and start reading B. When it reaches B's #Import of C, it will suspend the processing of B

```
1  #Import Prelude.qpl
2
3  toQubitList::(n:Int ;                  //Input: a probabilistic int
4              nq:List(Qubit), n:Int)= //Output: qubit list, original int
5  { use n in
6    { if n == 0      =>
7             { nq        = Nil}
8      (n mod 2) == 0 =>
9             { n'        = n >> 1;
10              (nq', n') = toQubitList(n');
11              nq        = Cons(|0>, nq') }
12     else => { n'       = n >> 1;
13              toQubitList(n'; nq',n');
14              nq        = Cons(|1>, nq') };
15     n = n //Recreate as probabilistic int
16   }
17 }
```

Figure 4.3: L-QPL function to create a qubit register

and read C. After completing the read of C, the compiler reverts to processing B. After completing the read of B, the compiler does a final reversion and finishes processing A. However, when A's #Import of C is reached, the compiler will ignore this import as it keeps track of the fact C has already been read.

In the signature on lines 3-4, the function accepts one quantum parameter of type **Int**. It returns a **Qubit** list and an **Int**. The integer returned, in this program, is computed to have the same value as the one passed in. If this had not been specified in this way, *the integer would have been destroyed by the function*. Generally, any usage of a quantum variable destroys that variable.

In the body of the function, note the **use** n in the block of statements. This allows repeated use of the variable n at lines 6, 8, 9, 12 and 15. In these uses, n is a classical variable, no longer on the quantum stack. The last usage on line 15 where n is assigned to itself, returns n to the quantum world.

4.3 L-QPL statements

The L-QPL language has the following statements:

Assignment: The assign statement, e.g. x = t;

Classical control: The **if** - **else** statement.

Case: The **case** for operating on constructed data types.

Measure: The **measure** statement which measures a qubit and executes dependent statements.

Use: The **use** and classical assign statements which operate on classical data, moving it on to the classical stack for processing.

Function calls: The various ways of calling functions or applying transformations.

Blocks: A group of statements enclosed by '{' and '}'.

Quantum control: Control of statements by the $\Leftarrow$ qualifier.

Divergence: The **zero** statement.

Other: the **discard** statement.

Most of these correspond to the conceptual statements for quantum pseudo-code as given in [Kni96].

4.3.1 Assignment statement

Assignments create variables. Typical examples of these are:

```
q1  = |0>;
i   = 42;
bt1 = Br(Leaf(q1), Br(Leaf(|0>), Leaf(|1>)));
```

Here the first line creates a qubit, q1, with initial value $|0\rangle$. The second line creates an integer ,i, with the value 42. The last line creates a binary tree, bt1, with q1 as its leftmost node, and the right node being a sub-tree with values $|0\rangle$ and $|1\rangle$ in the left and right nodes respectively. Note that after the execution of the third statement, the variable q1 is no longer in scope so the name may be reused by reassigning some other value to it.

The variables on the left hand side of an assignment are always quantum variables.

Syntax of assignment statements. An assignment statement always begins with an *identifier*. This must be followed by a single equals sign and then an *expression*. Expressions are introduced and defined in section 4.4 on page 83

Identifiers in L-QPL must always start with a lower case letter.

4.3.2 Classical control

Classical control provides a way to choose sets of instructions to execute based upon the values on the classical stack.

```
smallPrime::(a:Int | ;
             isSmallPrime:Bool) =
{ if a =< 1 => {isSmallPrime = false}
     a == 2 || a == 3 || a == 5  =>
                 {isSmallPrime = true}
     a == 4 => {isSmallPrime = false}
  else      => {isSmallPrime = false}
}
```

Figure 4.4: L-QPL program demonstrating **if–else**

The expressions in the selectors *must* be classical. This means they can only consist of operations on constants and classical identifiers. It is a semantic error to have an expression that depends on a quantum variable. For an example, see the code to determine if an input number is a small prime in figure 4.4 on the preceding page.

Some form of classical control is encountered in all current quantum programming languages. It is central to the semantics of [Sel04]. Note that many languages also include a classically controlled looping statement (such as while or do). L-QPL does not, relying on recursive functions to achieve the same end.

Syntax of the if − else statement. The statement starts with the word *if*, followed by one or more *selectors*. Each selector is composed of a classical Boolean expression e_b, the symbols => and a dependent block. The statement is completed by a special selector where the Boolean expression is replaced with the word *else*.

In the list of $e_i \Rightarrow b_i$ selectors, b_i is executed only when e_i is the first expression to evaluate to true. All others are skipped. The final grouping of **else** => *block* is a default and will be executed when all the selector expressions in a list evaluate to false.

When writing the dependent blocks of the selectors, quantum variable creation must be the same in each block. The compiler will give you a semantic warning if a quantum variable is created in one branch and not another.

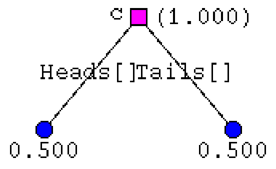
4.3.3 Measure statement

The **measure** statement performs a measurement of a qubit and executes code depending on the outcome. Currently in L-QPL, all measures are done with respect to the basis $\{|0\rangle, |1\rangle\}$. Referring to figure 4.5 on the next page, there is a **measure** on line 5.

Consider the program in figure 4.5 which emulates a coin flip. In the function cflip, a qubit is prepared by initializing it to $|0\rangle$ and applying the Hadamard transform. This

```
qdata Coin = {Heads | Tails}
cflip::( ; c:Coin) =
{   q = |0>;
    Had q;
    measure q of
       |0> => {c = Heads}
       |1> => {c = Tails}
}
main::() =
{   c = cflip()}
```

(a) Coin flip code



(b) Stack machine state at end

Figure 4.5: L-QPL program to do a coin flip

creates a qubit whose density matrix is $\begin{pmatrix} .5 & .5 \\ .5 & .5 \end{pmatrix}$. When this qubit is measured, it has a 50% chance of being 0 and an equal chance of being 1.

In the branches of the measure, different values are assigned to the return variable c. Each of these assignments happens with a probability of 50%. Once the measure statement is completed, the variable c will be Heads and Tails each with a probability of 50%. In the quantum stack machine this is represented as in sub-figure b of figure 4.5.

This illustrates the largest difference between quantum and classical processing of choices. In classical programming languages, a choice such as a case type statement *will only execute the code on one of the branches of the case*. In L-QPL, *every* branch may be executed.

```
main::() =
{   q = |0>;
    Had q;
    measure q of
       |0> => {i = 0}
       |1> => {i = 17};
    q = |0>
}
```

(a) Balanced creation

```
main::() =
{   q = |0>;
    Had q;
    measure q of
       |0> => {c = 0}
       |1> => {d = 17};
    q = |0>
}
```

(b) Unbalanced creation

Figure 4.6: L-QPL programs contrasting creation

When writing the dependent blocks of **measure** (and **case** in sub-section 4.3.4) variable creation must be the same in each dependent list of statements. The compiler will give you a semantic warning if a variable is created in one branch and not another.

For example, consider figure 4.6 on the preceding page. In the left hand program on the **measure** starting at line 4, each branch creates a variable named 'i'. This is legal and from line 7 forward, 'i' will be available.

On the other hand, the measure in the right hand program starting in line 4 assigns to the variable 'c' in the $|0\rangle$ branch and 'd' in the $|1\rangle$ branch. At line 7, neither variable will be available. The compiler will give the warnings:

```
Warning:  Unbalanced creation, discarding c of type INT
Warning:  Unbalanced creation, discarding d of type INT
```

Syntax of the measure statement. This statement starts with the word *measure*, followed by a variable name, which must be of type **Qubit**. Next, the keyword *of* signals the start of the two case selections. The case selection starts with either $|0\rangle$ or $|1\rangle$, followed by => and the block of dependent statements.

Note that *both* case selections for a qubit must be present. However, it is permissible to not have any statements in a block.

4.3.4 Case statement

The **case** statement is used with any variable of a declared datatype.

In figure 4.7 on the next page, the program declares the List data type, which is parametrized by one type variable and has two constructors: Nil which has no arguments and Cons which takes two arguments of types a and List(a) respectively.

The function reverse takes a list as an input argument and returns a single list, which is the original list in reverse order. Because of the linearity of the language the original

```
1  qdata List a = {Nil | Cons(a, List(a))}
2
3  reverse::(lis:List(a) ; revlis:List(a))=
4  {  rev' (lis, Nil ; revlis) }
5
6  rev'::(lis:List(a), accumIn:List(a) ; returnList:List(a))=
7  {  case lis of
8       Nil            => { returnList = accumIn}
9       Cons(hd, tail) => { acc        = Cons(hd, accumIn);
10                          returnList = rev'(tail, acc)}
11 }
```

Figure 4.7: L-QPL program demonstrating case, a function to reverse a list.

input list is not in scope at the end of the function. The function reverse delegates to the function rev′ which uses an accumulator to hold the list as it is reversed.

The case statement begins on line 7. For Nil, it assigns the accumulator to the return list. For Cons, it first adds the current element to the front of the accumulator list, then it uses a recursive call to reverse the tail of the original list with the new accumulator.

```
qdata TTree a = {Tip | Br(TTree(a), a, TTree(a)) | Node(a)}

treeMaxDepth::(t:TTree(a); depth:Int) =
{  case t of
     Tip     => {depth = 0}
     Node(_) => {depth = 1}
     Br(t1,_,t2) =>
               { j := treeMaxDepth(t1);
                 k := treeMaxDepth(t2);
                 if j > k => { depth = 1 + j}
                 else     => { depth = 1 + k}}
}
```

Figure 4.8: L-QPL program demonstrating case, a function to compute the max tree depth.

Consider the example in figure 4.8. TTree is a parametrized data type which depends on the type variable a. It has three constructors: Tip which takes no arguments; Br which takes three arguments of types TTree a, a and TTree a; and Node which takes one argument of type a.

In treeMaxDepth, the case statement on line 4 illustrates a "don't care" pattern for both the Node and Br constructors. This function returns the maximum depth of the TTree and actually discards the actual data elements stored at nodes.

Syntax of the case statement. This statement starts with the word *case*, followed by a variable of some declared type. Next, the keyword *of* signals the start of the case selections. The number of constructors in a type determine how many case selections the statement has. There is one selection for each constructor. Each case selection consists of a *constructor pattern*, a '=>' and dependent statements.

Constructor patterns are the constructor followed by a parenthesized list of variables and / or *don't care* symbols, '_'. Non-parametrized constructors appear without a list of variable names. The don't care symbol causes data to be discarded.

4.3.5 Use and classical assignment statements

The **use** statement is used with any variable of type **Int** or **Bool**. This statement has a single set of dependent statements. These may either be explicitly attached to the **use** statement or implicit. Implicit dependent statements are all the statements following the **use** until the end of the current block.

Classical assignment is grouped here as it is syntactic sugar for a **use** with implicit statements. This is illustrated in figure 4.9.

```
  ⋮               ⋮
                  i = exp;
i := exp;    ≡    use i;
s1;               s1;
  ⋮               ⋮
```

Figure 4.9: Syntactic sugar for **use** / classical assignment

The three types of classical use are semantically equivalent, but do have different syntaxes as illustrated in figure 4.10.

```
1 Br(t1,_,t2) =>
2   { j = treeMaxDepth(t1);
3     k = treeMaxDepth(t2);
4     use j,k in
5       { if j > k => { depth = 1 + j}
6         else      => { depth = 1 + k} }
7   }
```

(a) Explicit dependence

```
1 Br(t1,_,t2) =>
2   {  j = treeMaxDepth(t1);
3      k = treeMaxDepth(t2);
4      use j,k;
5      if j > k => { depth = 1 + j}
6      else     => { depth = 1 + k}
7   }
```

(b) Implicit dependence

```
1 Br(t1,_,t2) =>
2   { j  := treeMaxDepth(t1);
3     k  := treeMaxDepth(t2);
4     if j > k => { depth = 1 + j}
5     else     => { depth = 1 + k}
6   }
```

(c) Classical assign

Figure 4.10: Fragments of L-QPL programs contrasting **use** syntax

In sub-figure (a) of figure 4.10, the **use** statement starts on line 4. The next two statements are explicitly in its scope, which ends at line 6. In sub-figure (b) of the same figure, the **use** at line 4 is implicit. Its scope extends to line 7. Finally, in sub-figure(c), the same effect is achieved with two classical assignments at lines 2 and 3. The scope of these assignments extend to line 6.

Unlike data types and **Qubit**s, which have a maximum number of sub-stacks, an **Int** has the potential to have an unbounded number of values and therefore sub-stacks. The dependent statements of the **use** statement are executed for *each* of these values.

To execute different statements depending on the value of the **Int**, L-QPL provides the **if** - **else** statement as discussed in sub-section 4.3.2 on page 70. Typical use would be immediately after the **use** statement, or as the first statement of the dependent block of an explicit **use** statement.

Syntax of the use statement. This statement starts with the word **use**, followed by a list of variable names, which must be of type **Int** or **Bool**. If there is an explicit dependent block for the statement, it is given by the keyword **in** followed by the dependent block.

When the **use** statement is *not* followed by a dependent block, the rest of the statements in the enclosing block are considered in the scope of the **use**.

Classical assign syntax is a variable name, followed by the symbol ':=' followed by an expression. The expression must have type **Int** or **Bool**.

4.3.6 Function calls

Function calls include calling functions defined in programs and the predefined transforms. The list of predefined transforms valid in a L-QPL program are given in table 4.1 on the next page.

In addition to the predefined transforms, L-QPL allows you to prefix any of the predefined transformations with the string Inv− to get the inverse transformation. Controlled versions of all of these are available by using the quantum control construction as defined in sub-section 4.3.8 on page 81. For example, the **Toffoli3** gate as shown in table 2.1 on page 17 is simply a controlled-controlled-Not transformation.

The signatures of transforms are dependent upon the size of the associated matrix. A 2×2 matrix gives rise to the signature (q:**Qubit** ; q:**Qubit**). In general, a $2^n \times 2^n$ matrix will require n qubits in and out. The parametrized transforms such as **Rot** will require one or more integers as input.

Syntax of function calls

There are three different calling syntaxes for functions:

1. *Functional* — $(y_1,\ldots,y_m) = f(n_1,\ldots,n_k \,|\, x_1,\ldots,x_j)$.

L-QPL	**A.K.A.**	**Matrix**
Not	X, Pauli-X, ρ_X	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
RhoY	Y, Pauli-Y, ρ_Y	$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
RhoZ (=**Rot**(0))	Z, Pauli-Z, ρ_Z	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Had	Had, H	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
Swap	Swap	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
Phase (=**Rot**(2))	S, Phase $\sqrt{Z}$	$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
T(=**Rot**(3))	T, $\frac{\pi}{8}$, $\sqrt{S}$	$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$
Rot(n)	R_n, Rotation	$\begin{bmatrix} 1 & 0 \\ 0 & e^{2i\pi/2^n} \end{bmatrix}$

Table 4.1: L-QPL transforms

2. *Procedural* — $f(n_1, \ldots, n_k \,|\, x_1, \ldots, x_j \,;\, y_1, \ldots, y_m)$.

3. *Transformational* — $f(n_1, \ldots, n_k)\; z_1\; z_2\; \ldots\; z_j$.

Functions with classical and quantum inputs. These functions may be called[1] in each of the three ways.

```
    f ::(c1:Int,c2:Int | q1:Qubit, i1:Int ; a:Qubit, b:Int) = {...}
    ...
      (a,b) = f(c1,c2 | q1,i2);
      f(c1,c2 | q1,i2; a,b);
      f(c1,c2) q i;
```

When a function is called in the transformational syntax, as on the last line of the above code, the arguments separated by spaces (`q i` in the example) are both passed into the function as arguments and used as return variables. The arguments in the parenthesis (`c1,c2` in our example), *must* be classical and a semantic error will result if a quantum variable is used. If the number of in and out quantum arguments are not the same or their types do not match, this syntax is not available.

Functions which have no quantum input arguments. Functions which have only classical inputs may be called in either the functional or procedural syntax. As there are no input quantum arguments, transformational syntax is not allowed.

```
    g :: (c1:Int, c2:Int | ; r:Int, d:Int) = {...}
    ...
     (a,b) = g(c1,c2 |);
     g(c1,c2 | ; a,b);
```

[1] A function call where a single unparenthesized variable appears on the left hand side of the equals is actually an assignment statement. The right hand side of the assignment is a function expression. See sub-section 4.3.1 and sub-section 4.4.4 for further details.

Functions which have no classical input arguments. Functions having only quantum inputs may use all three syntaxes. In this case, where the classical variable list of arguments is empty, the "|" may be eliminated in procedural or functional calling, and the parenthesis may be eliminated in transformational calling.

```
h :: (q1:Int, q2:Int ; r:Int, d:Int) = {...}
...
  (a,b) = h(|c,d);
  (a,b) = h(c,d);
  h( | a,b ; c,d);
  h(a,b; c,d);
  h a b;
```

Linearity of function call arguments. Each input argument is no longer in scope after the function call. If the input is a simple identifier, the same identifier may be used in the output list of the function. The transformational syntax uses this technique to leave the variable names unchanged.

Syntactic forms of function and transform calls. In all three of the forms for function calling, the number and type of input arguments must agree with the definition of the function. Output identifiers must agree in number and their type is set according to the definition of the functions output parameters. Output variables are always quantum.

The *functional* syntax for function calling has three parts. The first part is a parenthesized list of variable names separated by commas. The parenthesized list is then followed by an equals sign. The right hand side consists of the function name followed by the parenthesized input arguments. The input arguments consists of two lists of arguments separated by '|'. The first list consists of the classical arguments, the second of the quantum arguments. Each argument must be a valid expression as defined in section 4.4 on page 83. If there are no classical arguments, the '|' is optional.

The *procedural* syntax for function calling starts with the function name, followed by a parenthesized grouping of input and output arguments. As in the functional form, the list of classical input arguments are separated from the quantum ones by '|', which may be eliminated when there are no classical arguments. The input arguments are separated from the output arguments by ';'.

The *transformational* syntax starts with the function name followed by a parenthesized list of classical expressions and then by a series of identifiers, separated by white space. A requirement for using this syntax is that the number and types of the input and output quantum arguments must be the same. The identifiers will be passed as input to the function and will be returned by the function. The parenthesis for the list of classical expressions may be eliminated when there are no classical parameters.

Function calls may also be expressions, which is discussed in sub-section 4.4.4.

4.3.7 Blocks

Blocks are created by surrounding a list of statements with braces. A block may appear wherever a statement does. All of the "grouping" types of statements require blocks rather than statements as their group. See, for example, the discussion on **case** statements in sub-section 4.3.4.

4.3.8 Quantum control

Quantum control provides a general way to create and use controlled unitary transforms in an L-QPL program. An example of quantum control is shown in the prepare and teleport functions in figure 4.11 on the next page.

```
prepare::( ; a:Qubit, b:Qubit)=
{ a = |0>;  b = |0>;
  Had a;
  Not b  <=  a;
}
teleport::(n:Qubit, a:Qubit, b:Qubit ; b:Qubit) =
{ Not a  <=  n ;
  Had n;
  measure a of
      |0> => {}  |1> => {Not b};
  measure n of
      |0> => {}  |1> => {RhoZ b}
}
```

Figure 4.11: L-QPL program demonstrating quantum control

Syntax of quantum control In L-QPL any statement, including block statements and procedure calls may be quantum controlled. Semantically, this will affect all transforms that occur within the controlled statement, including any of the transforms in a called function. The syntax is *statement* followed by the symbols <=, followed by a list of identifiers. These identifiers may be of any type, but are typically either qubits or constructed data types with qubit elements, such as a List(**Qubit**). Additionally, any identifier may be preceeded by a tilde (~) to indicate 0−control. The default is 1−control.

The identifiers that are controlling a statement can not be used in the statement. Controlling identifiers are exempt from linearity constraints in that they remain in scope after the quantum control construction.

The effect of the control statement is that all qubits in the control list, or contained in items in that list are used to control all unitary transformations done in the controlled statement.

4.3.9 Divergence

To force the execution of a program to diverge, you can use the **zero** statement. This will set the probability of the values of a program to 0, and is interpreted as non-termination.

4.3.10 Discard

In some algorithms, such as in recursive functions when they process the initial cases of constructed data, the algorithm does not specify any action. In these cases, the programmer may need to examine the requirements of linearity with respect to any passed in parameters. Often, these will need to be explicitly discarded in base cases of such algorithms. This is done via the discard statement. An example may be seen in figure C.29 on page 168.

4.4 L-QPL expressions

Expressions in L-QPL are used in many of the statements discussed in section 4.3 on page 69. The four basic types of expressions are identifiers, constants, constructor expressions, and calling expressions. Arithmetic and logical combinations of classical constants and classical identifiers are allowed. As well, constructor and calling expressions often take lists of other expressions as arguments.

4.4.1 Constant expressions

The possible constant expressions in L-QPL are shown in table 4.2 on the following page. The category column in this table contains the word "Classical" when the constant may be used in arithmetic or Boolean expressions.

Expression	Type	Category
integer	**Int**	Classical
true	**Bool**	Classical
false	**Bool**	Classical
$\|0\rangle$	**Qubit**	Quantum
$\|1\rangle$	**Qubit**	Quantum

Table 4.2: Allowed constant expressions in L-QPL

4.4.2 Identifier expressions

These expressions are just the identifier name. While an identifier may be used wherever an expression is expected, the reverse is not true. As an example, in function calls, any of the input arguments may be expressions but the output arguments *must* be identifiers.

Identifier expressions may be either quantum or classical in nature. As discussed in sub-section 4.3.1 on page 69, an identifier is first created by an assignment statement. When initially created, the identifier is always quantum. Using it where a classical expression is required will result in an error. When it is desired to operate on an identifier classically, it must first be the object of a **use** statement. In all statements in the scope of that **use** statement the identifier will be considered classical. See subsection 4.3.5 on page 75 for further information and examples.

4.4.3 Constructor expressions

These expressions are used to create new instances of declared data types. Consider this sample fragment of code.

```
qdata TTree a = {Tip | Br(TTree(a), a, TTree(a)) | Node(a)}
qdata List a  = {Nil | Cons(a, List(a))}

qbtree  = Br(Tip, |1>, Br(Node(|0>), |1>, Node(|1>)));
intlist = reverse(Cons(5,Cons(4,Cons(3,Cons(2,Cons(1,Nil))))));
```

These statements create a tree as in figure 4.12 and the list $[1, 2, 3, 4, 5]$. Compare the logical representation of qbtree as in figure 4.12 versus how it is stored in the quantum stack machine, shown in figure 4.13. The assignment statement which creates qbtree

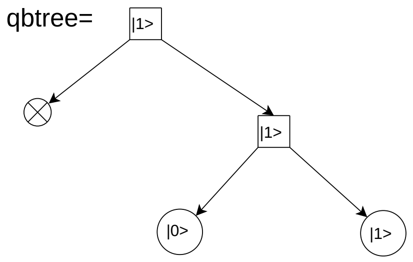


Figure 4.12: Pictorial representation of qbtree

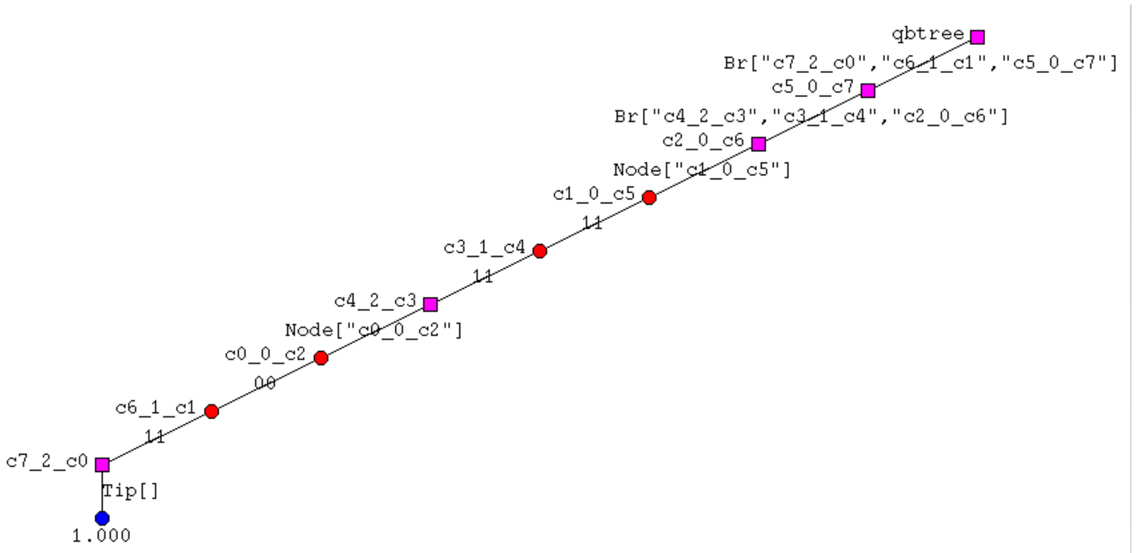


Figure 4.13: Quantum stack contents after creation of qbtree

uses five constructor expressions. The second assignment statement, which creates intlist uses six constructor expressions and one function expression.

Constructor expressions either have no arguments (e.g. Tip, Nil above), or require a parenthesized list of expressions which agree in both number and type with the template supplied at the declaration of the type. These expressions are unrestricted otherwise. They may be constants, identifiers, other constructor expressions, expression calls or compound expressions. Any expressions that are classical in nature, such as constants, are upgraded to quantum automatically.

4.4.4 Function expressions

When a function returns a single value, it may be used in a function expression. The bottom two lines of listing below shows two examples of function expressions.

```
f::(c1:Int, c2:Int, c3:Int | q1:Qubit, i1:Int ; out:Qubit)
= { ... }
...
    qout  = f(c1,c2,c3 | q1,i2);
    qlist = Cons(f(1,2,3 | qout,5),Nil);
```

In the first function expression, f is the right hand side of an assignment statement. The assignment statement creates the variable qout with the value returned by the function.

In the second function expression, f is the first argument of a constructor expression which will create a one element List(**Qubit**). The constructor expression is part of an assignment statement which creates the variable qlist and sets it to the one element list. Note that due to linearity, the variable qout is no longer available after the second function expression.

A function expression is always a quantum expression, so it may only be used in those places where quantum expressions are allowed. Nesting of these calls inside constructor expressions, other function expressions and function calls is allowed.

```
1 qdata List a = {Nil | Cons(a, List(a))}
2
3 append::(list1:List(a), list2:List(a); appendList:List(a))=
4 {   case list1 of
5       Nil => {appendList = list2}
6       Cons(hd, tail) =>
7         { appendList = Cons(hd, append(tail, list2))}
8 }
```

Figure 4.14: L-QPL code for appending two lists

In figure 4.14, line 7 shows the append being used as a function expression inside of a constructor expression.

Chapter 5

The quantum stack machine

5.1 Introduction to the quantum stack machine

The quantum stack machine provides an execution environment where quantum and classical data may be manipulated. The primary component of this machine is the *quantum stack*, which stores both quantum and probabilistic data.

The quantum stack has the same function as a classical stack in that it provides the basic operations and data structures required for quantum computation. Chapter 3 initially showed how quantum circuits can be interpreted as acting on a simple quantum stack consisting of bits and qubits. Later sections of chapter 3 extended the quantum stack with datatype and classical data nodes, together with operations on those nodes. Section 3.7 gave the interpretation of recursive functions acting on a quantum stack.

This chapter describes a machine using this full quantum stack and other data structures to provide an execution environment for L-QPL programs.

5.2 Quantum stack machine in stages

The quantum stack machine is described in terms of four progressively more elaborate stages. The first stage is the *basic QS-machine*, labelled BQSM. This stage provides facilities for the majority of operations of our machine, including classical operations, adding and discarding data and classical control. The second stage, the *labelled QS-machine*, called LBQSM adds the capability of applying unitary transforms with the modifiers `Left, Right` and `IdOnly` as introduced in section 3.4. The third stage, the *controlled QS-machine*, is labelled CQSM and provides the ability to do quantum control. The final stage, the *QS-machine*, is labelled QSM and adds the ability to call subroutines and do recursion.

These stages are ordered in terms of complexity and the operations definable on them. The ordering is:

$$\text{BQSM} < \text{LBQSM} < \text{CQSM} < \text{QSM}$$

When a function is defined on one of the lower stages, it is possible to lift it to a function on any of the higher stages.

5.2.1 Basic quantum stack machine

The quantum stack machine transitions for the quantum instructions are defined at this stage. The state of the basic quantum stack machine has a code stream, $\mathcal{C}$, a classical stack, S, a quantum stack, Q, a dump, D and a name supply, N.

$$(\mathcal{C}, S, Q, D, N) \tag{5.1}$$

The code, $\mathcal{C}$, is a list of machine instructions. Transitions effected by these instructions are detailed in sub-section 5.4.1 on page 96. English descriptions of the instructions and what they do are given in appendix B.1 on page 127.

The classical stack, S, is a standard stack whose items may be pushed or pulled onto the top of the stack and specific locations may be accessed for both reading and updating. Classical arithmetic and Boolean operations are done with the top elements of the classical stack. Thus, an add will pop the top two elements of the classical stack and then push the result on to the top of the stack.

The dump, D, is a holding area for intermediate results and returns. This is used when measuring quantum bits, using probabilistic data, splitting constructed data types and for calling subroutines. Further details are given in sub-section 5.2.6 on page 92.

The name supply, N, is an integer that is incremented each time it is used. The name supply is used when binding nodes to constructed data nodes. As they are bound, they are renamed to a unique name generated from the name supply. For further details on this, see the transitions for `QBind` at sub-section 5.4.2 on page 96.

5.2.2 Labelled quantum stack machine

The labelled QS-machine, designated as LBQSM, extends BQSM by labelling the quantum stack, $L(Q)$. The quantum stack is labelled to control the application of unitary transformations, which allows quantum control to be implemented.

The labelled QS-machines state is a tuple of five elements:

$$(\mathcal{C}, S, L(Q), D, N) \tag{5.2}$$

The quantum stack is labelled by one of four labels: `Full, Right, Left` or `IdOnly`. These labels describe how unitary transformations will be applied to the quantum stack.

When this labelling was introduced in section 3.4, it was used as an instruction modifier rather than a labelling of the quantum stack. While the implementation of these modifiers is changed, the effect on the quantum stack is the same. The quantum stack machine transitions for unitary transformations are detailed in sub-section 5.4.8 on page 107.

5.2.3 Controlled quantum stack machine

The controlled quantum stack machine, CQSM, adds the capability to add or remove quantum control. This stage adds a control stack, C, and changes the tuple of classical stack, labelled quantum stack, dump and name supply into a list of tuples of these elements. In the machine states, a list will be denoted by enclosing the list items or types in square brackets.

The CQSM state is a tuple of three elements, where the third element is a list of four-tuples:

$$(\mathcal{C}, C, [(S, L(Q), D, N)]) \tag{5.3}$$

The control stack is implemented using a list of functions, each of which is defined on the third element of CQSM. The functions in the control stack transform the list of tuples $(S, L(Q), D, N)$. Control points are added to the control stack by placing an identity function at the top of the stack. Control points are removed by taking the top of the control stack and applying it to the current third element of CQSM, resulting in a new list of tuples. Adding a qubit to control will modify the function on top of the control stack and change the list of tuples of $(S, L(Q), D, N)$.

5.2.4 The complete quantum stack machine

The complete machine, QSM, allows the implementation of subroutine calling. Its state consists of an infinite list of CQSM elements.

$$\text{Inflist}(\mathcal{C}, C, [(S, L(Q), D, N)]) \tag{5.4}$$

Subroutine calling is done in an iterative manner. At the head of the infinite list, no subroutines are called, but result in divergence. Divergence is represented in the quantum stack machine by a quantum stack with a trace of 0.

In the next position of the infinite list, a subroutine will be entered once. If the subroutine is recursive or calls other subroutines, those calls will diverge. The next position of the infinite list will call one more level. At the n^{th} position of the infinite list subroutines are executed to a call depth of n.

5.2.5 The classical stack

The machine uses and creates values on the classical stack when performing arithmetic operations. This object is a standard push-down stack with random access. Currently it accommodates both integer and Boolean values.

5.2.6 Representation of the dump

When processing various operations in the machine, such as those labelled as quantum control (measure et. al.), the machine will need to save intermediate stack states and results. To illustrate, when processing a case deconstruction of a datatype, the machine saves all partial trees of the node on the dump together with an empty stack to accumulate the results of processing these partial trees. After processing each case the current quantum stack is merged with the result stack and the next partial stack is processed. The classical stack is also saved in the dump element at the beginning of the process and reset to this saved value when each case is evaluated.

The dump is a list of *dump elements*. There are two distinct types of dump elements, one for quantum control instructions and one used for call statements. The details of these elements may be found in the description of the quantum control transitions in sub-section 5.4.5 on page 101 and function calling in sub-section 5.4.9 on page 109.

5.2.7 Name supply

The name supply is a read-only register of the machine. It provides a unique name when binding nodes to a data node. The implementation uses an integer value which is incremented for each of the variables in a selection pattern in a case statement. It is reset to zero at the start of each program.

5.3 Representation of data in the quantum stack

The quantum stack was introduced and described in chapter 3. This section will give further details of the implementation of the quantum stacks and show example nodes.

5.3.1 Representation of qubits.

A single qubit is represented on the quantum stack as a node with four possible branches. This assumes a basis for quantum computation of two elements, which is identified with $(0,1)$ and $(1,0)$ in $\mathbb{C}^2$. The four possible values of the branches represent the elements of the qubit's density matrix. This is illustrated in figure 5.1. From left to right, the branches are labelled with $00, 01, 10$ and 11. The value at each branch is .5. This corresponds to the density matrix $\begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$

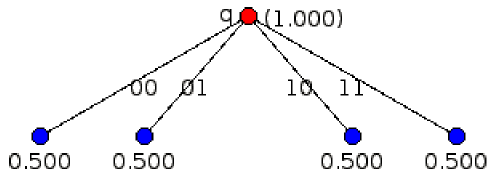


Figure 5.1: A qubit after a Hadamard transform

With multiple qubits, the representation becomes hierarchical. For example, two qubits will be represented by a tree with one of the qubits at the top and each of its sub-branches having the second qubit below it. Consider applying a Hadamard transform to one qubit, followed by a controlled-Not with that qubit as the control. This is a standard way to entangle two qubits. As illustrated in figure 5.2, this creates a tree in the quantum stack with a total of four non-zero leaves. The quantum stack in the figure corresponds to a sparse representation of the density matrix:

$$\left[\begin{array}{cc|cc} .5 & 0 & 0 & .5 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ .5 & 0 & 0 & .5 \end{array}\right]$$

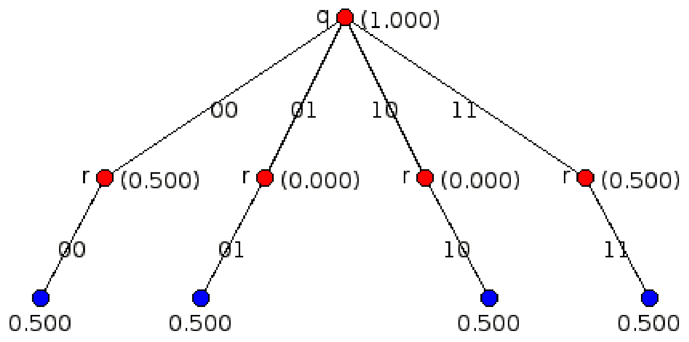


Figure 5.2: Two entangled qubits

5.3.2 Representation of integers and Boolean values

Numeric and Boolean data in the quantum stack machine is represented by a node with a sub-branch for each value that occurs with a non-zero probability. These values may be of either integer or Boolean type.

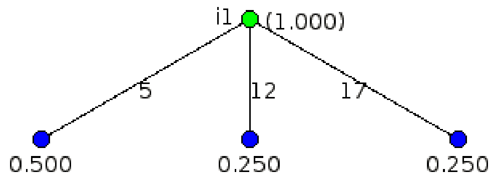


Figure 5.3: An integer with three distinct values

Figure 5.3 on the preceding page depicts an integer i1 which has a 50% probability of being 5, and 25% each of being 12 or 17.

5.3.3 Representation of general data types

The general datatype is represented as a node with one branch for each of the constructors that occurs with a non-zero probability. Each branch is labelled by the constructor and the names of any nodes that are bound to it[1]. These nodes will be referred to as *bound nodes*.

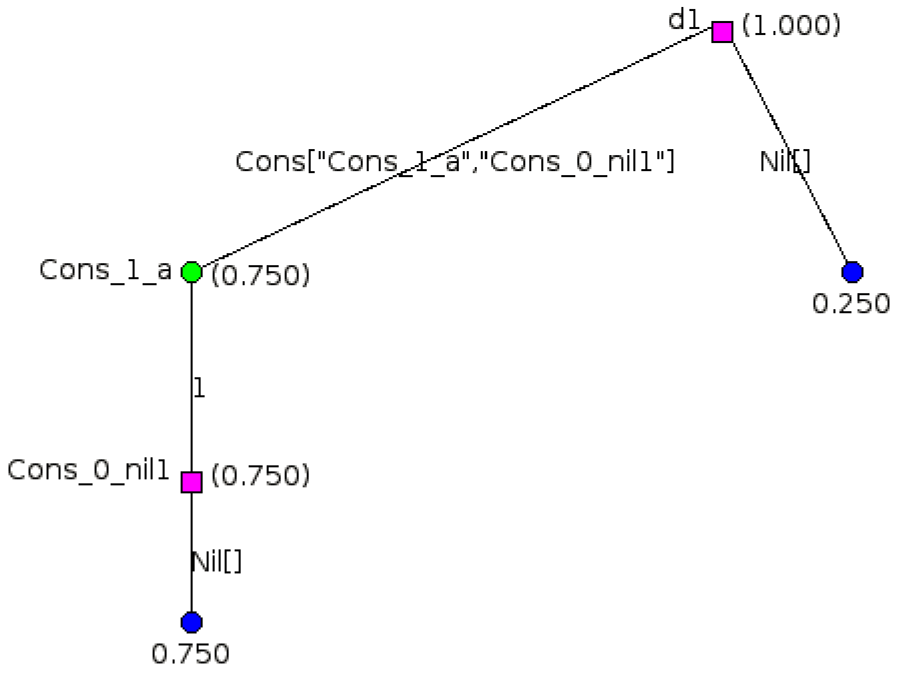


Figure 5.4: A list which is a mix of [ ] and [1].

For example, in the **List** that appears in figure 5.4, the top node is a mix of values. The node d1 has a 25% chance of being `Nil` and 75% of being a `Cons` of two bound nodes. The first bound node is an element of the base type, integer. It is labelled

[1]For example, in **List**s of integers, the `Cons` constructor requires a base integer and another **List**.

`Cons_1_a` which is an integer node having the single value 1. The second bound element is `Cons_0_nil1`, which is another list having the single value of `Nil`.

5.4 Quantum stack machine operation

This section describes the actual transitions of the stack machine for each of the instructions in the machine.

5.4.1 Machine transitions

The majority of the transitions presented in this section are defined on a machine of type $\mathrm{B}\textsc{qsm} = (\mathcal{C}, S, Q, D, N)$ as was introduced in sub-section 5.2.1 on page 89. As discussed in that section, labelling only affects the transition of unitary transforms. All other instruction transitions ignore it, giving us:

$$\mathit{Ins}(L(Q)) = L(\mathit{Ins}(Q))$$

where Ins is the transition of some other instruction.

The transition for the application of transformations will use $\mathrm{LB}\textsc{qsm}$, while the transition for the add/remove control instructions uses the machine state of $\mathrm{C}\textsc{qsm}$. The call instruction uses the state of the complete machine, QSM, which allows recursion. All of these stages and their associated states were defined and discussed in section 5.2 on page 88.

5.4.2 Node creation

There are three instructions which allow us to create data on the stack and one which binds sub-nodes into a data type. These are `QLoad, QCons, QMove` and `QBind`. The transitions are shown in figure 5.5 on the following page.

The instructions do the following tasks:

`QLoad` $\mathtt{nm}\ |\mathtt{i}\rangle$ — Load a new qubit named $\mathtt{nm}$ on top of the quantum stack with the value $|\mathtt{i}\rangle$;

`QCons` $\mathtt{nm}\ \mathtt{Cns}$ – Load a new datatype node on top of the quantum stack with name $\mathtt{nm}$ and value $\mathtt{Cns}$. Sub-nodes are not bound by this instruction.

`QMove` $\mathtt{nm}$ — Load a new classical node on top of the quantum stack with name $\mathtt{nm}$ and value taken from the top of the classical stack. If the classical stack is empty, the value is defaulted to 0.

`QBind` $\mathtt{nm}$ — Binds a sub-node down the branch of the node to the datatype constructor on top of the quantum stack. Furthermore, the act of binding will cause the newly bound sub-node to be renamed so that it is hidden until an unbind is performed. `QBind` uses the name supply, N, to create the new name for the sub-node. The machine will generate an exception if the top of the quantum stack is not a single branched datatype or if a node named $\mathtt{nm}$ is not found.

$$
\begin{aligned}
&(\text{QLoad } x\ |k\rangle : \mathcal{C}, S, Q, D, N)\\
&\qquad\qquad\qquad \Longrightarrow\ (\mathcal{C}, S, x{:}[|k\rangle \to Q], D, N)\\
&(\text{QCons } x\ c{:}\mathcal{C}, S, Q, D, N)\\
&\qquad\qquad\qquad \Longrightarrow\ (\mathcal{C}, S, x{:}[c\{\} \to Q], D, N)\\
&(\text{QMove } x{:}\mathcal{C}, v{:}S, Q, D, N)\\
&\qquad\qquad\qquad \Longrightarrow\ (\mathcal{C}, S, x{:}[\bar{v} \to Q], D, N)\\
&(\text{QBind } z_0{:}\mathcal{C}, S, x{:}[c\{z_1', \dots, z_n'\} \to Q], D, N)\\
&\qquad\qquad\qquad \Longrightarrow\ (\mathcal{C}, S, x{:}[c\{z(N), z_1', \dots, z_n'\} \to Q[z(N)/z_0]], D, N')
\end{aligned}
$$

Figure 5.5: Transitions for node construction

5.4.3 Node deletion

Three different instructions, `QDelete`, `QUnbind` and `QDiscard` remove data from the quantum stack. These instructions are the converses of `QBind`, `QLoad` and `QMove`. Their transitions are shown in figure 5.6 and figure 5.7. The instructions do the following tasks:

`QDelete` — removes the top node of the stack *and any bound sub-nodes*. This instruction has no restrictions on the number of sub-stacks or bindings in a data node;

`QDiscard` — removes the top node of the stack. In all cases, the top node can only be removed when it has a single sub-stack. For datatype nodes, `QDiscard` also requires there are no bound sub-nodes.

`QUnbind` nm — removes the first bound element from a data type *provided it has a single sub branch*. The datatype node must be on top of the quantum stack. The newly unbound sub-node is renamed to nm.

$$\begin{aligned}
&(\text{QDelete:}\mathcal{C}, S, Q{:}[|k_{ij}\rangle \to Q_{ij}], D, N) &&\Longrightarrow&& (\mathcal{C}, S, (Q_{00} + Q_{11}), D, N)\\
&(\text{QDelete:}\mathcal{C}, S, DT{:}[c_i\{b_{ij}\} \to Q_i], D, N) &&\Longrightarrow&& (\mathcal{C}, S, \textstyle\sum_i(\mathtt{del}(\{b_{ij}\}, Q_i)), D, N)\\
&(\text{QDelete:}\mathcal{C}, S, I{:}[\overline{v}_i \to Q_i], D, N) &&\Longrightarrow&& (\mathcal{C}, S, \textstyle\sum_i Q_i, D, N)
\end{aligned}$$

Figure 5.6: Transitions for destruction

For the `QDelete` instruction, the type of node is irrelevant. It will delete the node and, in the case of datatype nodes, any bound nodes. This instruction is required to implement sub-routines that have parametrized datatypes as input arguments. For example, the algorithm for determining the length of a list is to return 0 for the “Nil” constructor and add 1 to the length of the tail list in the “Cons” constructor. When doing this, the elements of the list are deleted due to the linearity of L-QPL. The compiler will have no way of determining the type of the elements in the list and therefore could not generate the appropriate quantum split and discards. The solution is to use a `QDelete` instead.

The subroutine $\mathtt{del}$ used in the transitions in figure 5.6 will recursively rotate up and then delete the bound nodes of a datatype node.

$$
\begin{aligned}
(\text{QDiscard}{:}\mathcal{C}, S, x{:}[|k\rangle \to Q], D, N) &\implies (\mathcal{C}, S, Q, D, N)\\
(\text{QDiscard}{:}\mathcal{C}, S, x{:}[c\{\} \to Q], D, N) &\implies (\mathcal{C}, S, Q, D, N)\\
(\text{QDiscard}{:}\mathcal{C}, S, x{:}[\overline{v} \to Q], D, N) &\implies (\mathcal{C}, v{:}S, Q, D, N)\\
(\text{QUnbind } y{:}\mathcal{C}, S, x{:}[c\{z_1', \dots, z_n'\} \to Q], D, N)&\\
&\implies (\mathcal{C}, S, x{:}[c\{z_2', \dots, z_n'\} \to Q[y/z_1']], D, N)
\end{aligned}
$$

Figure 5.7: Transitions for removal and unbinding

The renaming is an integral part of the `QUnbind` instruction, as a compiler will not be able to know the bound names of a particular data type node. The instruction does *not* delete the data type at the top of the stack or the unbound node. If the top node is not a data type or has more than a single branch or does not have any bound nodes, the machine will generate an exception.

The machine ensures that it does not create name capture issues by rotating the bound node to the top of the sub-stack before it does the rename. That is, given the situation as depicted in the transitions, the quantum stack machine performs the following operations:

1. $Q' \leftarrow \mathtt{pull}(z_1', Q)$;
2. $Q'' \leftarrow Q'[y/z_1']$;
3. z_1' is removed from the list of constructors;
4. The new quantum stack is now set to $x{:}[c\{z_2', \dots, z_n'\} \to Q'']$.

5.4.4 Stack manipulation

Most operations on a quantum stack affect only the top of the stack. Therefore, the machine must have ways to move items up the stack. This requirement is met by the instructions `QPullup` and `QName`. The transitions are shown in figure 5.8.

The instructions do the following tasks:

`QPullup` $\mathtt{nm}$ — brings the *first* node named $\mathtt{nm}$ to the top of the quantum stack. It is not an error to try pulling up a non-existent address. The original stack will not be changed in that case.

`QName` $\mathtt{nm}_1$ $\mathtt{nm}_2$ — renames the first node in the stack having $\mathtt{nm}_1$ to $\mathtt{nm}_2$.

$$\begin{aligned}
(\text{QPullup } x{:}\mathcal{C}, S, Q, D, N) &\quad\Longrightarrow\quad (\mathcal{C}, S, \mathsf{pull}(x, Q), D, N)\\
(\text{QName } x\ y{:}\mathcal{C}, S, Q, D, N) &\quad\Longrightarrow\quad (\mathcal{C}, S, Q[y/x], D, N)
\end{aligned}$$

Figure 5.8: Transitions for quantum stack manipulation

A `QPullup` $\mathtt{nm}$ has the potential to be an expensive operation as the node $\mathtt{nm}$ may be deep in the quantum stack . In practice, many pullups interact with only the top two or three elements of the quantum stack.

The algorithm for pullup is based on preserving the bag of *path signatures*. A path signature for a node consists of a bag of ordered pairs (consisting of the node name and the branch constructor) where every node from the top to the leaf is represented. Pulling up a node will reorder the sub-branches below nodes to keep this invariant.

Due to the way arguments of recursive subroutines are handled in L-QPL, it is actually possible to get multiple nodes with the same name, however, this does not cause a referencing problem as only the highest such node is actually available in the L-QPL program.

5.4.5 Measurement and choice

The instructions `Split`, `Measure` and `Use` start the task of operating on a node's partial stacks, while the fourth, `EndQC` is used to iterate through the partial stacks. The transitions are shown in figure 5.9 on the following page.

The instructions do the following tasks:

`Use` $\mathtt{Lbl}$ — uses the classical node at the top of the quantum stack and executes the code at $\mathtt{Lbl}$ for each of its values.

`Split` $(\mathtt{c}_1, \mathtt{lbl}_1), \ldots, (\mathtt{c}_n, \mathtt{lbl}_n)$ — uses the datatype node at the top of the stack and execute a jump to the code at $\mathtt{lbl}_i$ when there is a branch having constructor $\mathtt{c}_i$. Any constructors not mentioned in the instruction are removed from the node first. There is no ordering requirement on the pairs of constructors and labels in `Split`.

`Measure` $\mathtt{Lbl}_{00}$ $\mathtt{Lbl}_{11}$ — using the qubit node on top of the quantum stack, executes the code at its two labels for the 00 and 11 branches. The off-diagonal elements of the qubit will be discarded. This implements a non-destructive measure of the qubit.

`EndQC` — signals the end of processing of dependent instructions and begins processing the next partial stack. When all values are processed, merges the results and returns to the instruction after the corresponding `Measure`, `Use` or `Split` instruction.

Each of the code fragments pointed to by the instruction labels *must* end with the instruction `EndQC`. The `EndQC` instruction will trigger execution of the code associated with the next partial stack.

$$(\text{Use } \triangleright\mathcal{C}_U{:}\mathcal{C}, S, x{:}[\bar{v}_i \to Q_i], D, N)$$
$$\Longrightarrow\ (\text{EndQC}, S, 0, \mathcal{Q}c(S, [(x_i{:}v_i \to Q_i, \triangleright\mathcal{C}_U)], \triangleright\mathcal{C}, 0){:}D, N)$$

$$(\text{EndQC}, S', Q, \mathcal{Q}c(S, [(x_i{:}v_i \to Q_i, \triangleright\mathcal{C}_U)]_{i=j,\ldots,m}, \triangleright\mathcal{C}, Q'){:}D, N)$$
$$\Longrightarrow\ (\mathcal{C}_U, S, x_j, \mathcal{Q}c(S, [(x_i{:}v_i \to Q_i, \triangleright\mathcal{C}_U)]_{i=j+1,\ldots,m}, \triangleright\mathcal{C}, Q+Q'){:}D, N)$$

$$(\text{EndQC}, S', Q, \mathcal{Q}c(S, [], \triangleright\mathcal{C}, Q'){:}D, N)$$
$$\Longrightarrow\ (\mathcal{C}, S, Q+Q', D, N)$$

$$(\text{Split } [(c_i, \triangleright\mathcal{C}_i)]{:}\mathcal{C}, S, x{:}[c_i\{V_i\} \to Q_i], D, N)$$
$$\Longrightarrow\ (\text{EndQC}, S, 0, \mathcal{Q}c(S, [(x_i{:}c_i\{V_i\} \to Q_i, \triangleright\mathcal{C}_i)], \triangleright\mathcal{C}, 0){:}D, N)$$

$$(\text{EndQC}, S', Q, \mathcal{Q}c(S, [(x_i{:}c_i\{V_i\} \to Q_i, \triangleright\mathcal{C}_i)]_{i=j,\ldots,m}, \triangleright\mathcal{C}, Q'){:}D, N)$$
$$\Longrightarrow\ (\mathcal{C}_j, S, x_j, \mathcal{Q}c(S, [(x_i{:}c_i\{V_i\} \to Q_i, \triangleright\mathcal{C}_i)]_{i=j+1,\ldots,m}, \triangleright\mathcal{C}, Q+Q'){:}D, N)$$

$$(\text{EndQC}, S', Q, \mathcal{Q}c(S, [], \triangleright\mathcal{C}, Q'){:}D, N)$$
$$\Longrightarrow\ (\mathcal{C}, S, Q+Q', D, N)$$

$$(\text{Meas } \triangleright\mathcal{C}_0\ \triangleright\mathcal{C}_1{:}\mathcal{C}, S, x{:}[|0\rangle \to Q_0, |1\rangle \to Q_1, \ldots], D, N)$$
$$\Longrightarrow\ (\text{EndQC}, S, 0, \mathcal{Q}c(S, [(x_k{:}\,|k\rangle \to Q_k, \triangleright\mathcal{C}_k)]_{k\in\{0,1\}}, \triangleright\mathcal{C}, 0){:}D, N)$$

$$(\text{EndQC}, S', Q, \mathcal{Q}c(S, [(x_k{:}\,|k\rangle \to Q_k, \triangleright\mathcal{C}_k)]_{k\in\{0,1\}}, \triangleright\mathcal{C}, Q'){:}D, N)$$
$$\Longrightarrow\ (\mathcal{C}_0, S, x_0, \mathcal{Q}c(S, [(x_1{:}\,|1\rangle \to Q_1, \triangleright\mathcal{C}_1)], \triangleright\mathcal{C}, Q+Q'){:}D, N)$$

$$(\text{EndQC}, S', Q, \mathcal{Q}c(S, [(x_1{:}\,|1\rangle \to Q_1, \triangleright\mathcal{C}_1)], \triangleright\mathcal{C}, Q'){:}D, N)$$
$$\Longrightarrow\ (\mathcal{C}_1, S, x_1, \mathcal{Q}c(S, [], \triangleright\mathcal{C}, Q+Q'){:}D, N)$$

$$(\text{EndQC}, S', Q, \mathcal{Q}c(S, [], \triangleright\mathcal{C}, Q'){:}D, N)$$
$$\Longrightarrow\ (\mathcal{C}, S, Q+Q', D, N)$$

Figure 5.9: Transitions for quantum node choices

The `QUnbind` is meant to be used at the start of the dependent code of a `Split` instruction. The sequencing to process a datatype node is to do a `Split`, then in each of the dependent blocks, execute `QUnbind` instructions, possibly interspersed with `QDelete` instructions when the bound node is not further used in the code. This is always concluded with a `QDiscard` that discards the data node which was the target of the `Split`.

In the following discussion there are no significant differences between the `Split` and `Measure`. The action of `Split` is described in detail.

The `Split`, `Measure` and `Use` instructions make use of the dump. The dump element used by these instructions consists of four parts:

- *The return label.* This is used when the control group is complete.
- *The remaining partial stacks.* A list consisting of pairs of quantum stacks and their corresponding label. These partial stacks are the ones waiting to be processed by the control group.
- *The result quantum stack.* This quantum stack accumulates the merge result of processing each of the control groups partial stacks. This is initialized to an empty stack with a zero trace.
- *The saved classical stack.* The classical stack is reset to this value at the start of processing a partial stack and at the end. This occurs each time an `EndQC` instruction is executed.

The instruction `Split` $[(c1, l1), (c2, l2)]$ begins with the creation of a dump entry holding $[(c1 \to Q_1, l1), (c2 \to Q_2, l2)]$ as the list of partial stacks and label pairs. The dump entry will hold 0 quantum stack as the result stack, the current state of the classical stack and the address of the instruction following the `Split`. The final processing of the `Split` instruction sets the current quantum stack to zero and sets the next code to be executed to be `EndQC`.

The `EndQC` will change the top dump element by removing the first pair $(c1 \to Q_1, l1)$ from the execution list. It will set the current quantum stack to the first element of this pair and the code pointer to the second element. Execution then proceeds with the first instruction at $l1$.

When the next `EndQC` instruction is executed, the dump will again be changed. First the current quantum stack will be merged with the result stack on the dump. Then the next pair of partial quantum stack P_q $(= c2 \to Q_2)$ and code pointer $l2$ is removed from the execution list. The current quantum stack is set to P_q and the code pointer is set to $l2$. Finally, the classical stack is reset to the one saved in the dump element.

When the partial stack list on the dump element is empty, the `EndQC` instruction will merge the current quantum stack with the result stack and then set the current quantum stack to that result. The classical stack is reset to the one saved on the dump, the code pointer is set to the return location saved in the dump element and the dump element is removed. Program execution then continues from the saved return point.

Normally, the first few instructions pointed to by the $Label$ in the pairs of constructor and code labels will unbind any bound nodes and delete the node at the top of the stack. QSM does not *require* this, hence, it is possible to implement both destructive and non-destructive measurements and data type deconstruction.

Using classical values. The `Use` instruction introduced above differs from the instructions `Split` and `Measure` in that it works on a node that may an unbounded number of sub-nodes. The `Use` lbl instruction moves all the partial stacks to the quantum stack, one at a time, and then executes the code at lbl for the resulting machine states. Normally, this code will start with a QDiscard, which will put the node value for that partial stack onto the classical stack, and finish with an EndQC to trigger the processing of the next partial stack.

The dump and EndQC processing for a Use lbl is the same as for a Split or Measure. The execution list pairs will all have the same label, the lbl on the instruction.

5.4.6 Classical control

The machine provides the three instructions Jump, CondJump and NoOp for branch control. Jumps are allowed only in a forward direction. The transitions for these are shown in figure 5.10. The instructions do the following tasks:

Jump lbl — causes execution to continue with the code at lbl.

CondJump lbl — examines the top of the classical stack. When it is False, execution will continue with the code at lbl. If it is any other value, execution continues with the instruction following the CondJump.

NoOp — does nothing in the machine. Execution continues with the instruction following the NoOp.

$$
\begin{array}{lcl}
(\text{Jump} \rhd \mathcal{C}_J{:}\mathcal{C}, S, Q, D, N) & \Longrightarrow & (\mathcal{C}_J, S, Q, D, N) \\
(\text{CondJump} \rhd \mathcal{C}_J{:}\mathcal{C}, \texttt{False}{:}S, Q, D, N) & \Longrightarrow & (\mathcal{C}_J, S, Q, D, N) \\
(\text{CondJump} \rhd \mathcal{C}_J{:}\mathcal{C}, \texttt{True}{:}S, Q, D, N) & \Longrightarrow & (\mathcal{C}, S, Q, D, N) \\
(\text{NoOp}{:}\mathcal{C}, S, Q, D, N) & \Longrightarrow & (\mathcal{C}, S, Q, D, N)
\end{array}
$$

Figure 5.10: Transitions for classical control.

No changes are made to the classical stack, the quantum stack or the dump by these instructions. While `NoOp` does nothing, it is allowed as the target of a jump. This is used by the L-QPL compiler in the code generation as the instruction following a `Call`.

5.4.7 Operations on the classical stack

The machine has five instructions that affect the classical stack directly. They are `CGet`, `CPut`, `CPop`, `CApply` and `CLoad`, with transitions shown in figure 5.11. The instructions perform the following tasks:

`CPop` — destructively removes the top element of the classical stack.

`CGet` n — copies the n^{th} element of the classical stack to the top of the classical stack.

`CApply` op — applies the operation op to the top elements of the classical stack, replacing them with the result of the operation. Typically, the op is a binary operation such as *add*.

`CLoad` v — places the constant v on top of the classical stack.

$$
\begin{array}{lcl}
(\text{CPop}{:}\mathcal{C}, v{:}S, Q, D, N) & \Longrightarrow & (\mathcal{C}, S, Q, D, N) \\
(\text{CGet } n{:}\mathcal{C}, v_1{:}\cdots{:}v_n{:}S, Q, D, N) & \Longrightarrow & (c, v_n{:}v_1{:}\cdots{:}v_n{:}S, Q, D, N) \\
(\text{CPut } n{:}\mathcal{C}, v_1{:}\cdots{:}v_n{:}S, Q, D, N) & \Longrightarrow & (c, v_1{:}\cdots{:}v_1{:}S, Q, D, N) \\
(\text{CApply } \mathsf{op}_n{:}\mathcal{C}, v_1{:}\cdots{:}v_n{:}S, Q, D, N) & \Longrightarrow & (\mathcal{C}, \mathsf{op}_n(v_1, \ldots, v_n){:}S, Q, D, N) \\
(\text{CLoad } n\ {:}\mathcal{C}, S, Q, D, N) & \Longrightarrow & (\mathcal{C}, n{:}S, Q, D, N)
\end{array}
$$

Figure 5.11: Transitions for classical stack operations.

5.4.8 Unitary transformations and quantum control

The QS-Machine has three instructions which add or remove qubits (and other nodes) from quantum control. The instruction transitions in this group are defined directly on CQSM or LBQSM, as they will either affect the control stack (`AddCtrl, QCtrl, UnCtrl`) or need to take into account the labelling of the quantum stacks (`QApply`).

The first three instructions do not affect the actual state of the quantum stack, classical stack or dump. The `QApply` does affect the state of the quantum stack. The transitions are shown in figure 5.12 on the following page.

The instructions perform the following tasks:

`AddCtrl` — starts a new control point on the control stack.

`QCtrl` — adds the node at the top of the quantum stack, together with any dependent sub-nodes to the control stack.

`UnCtrl` — removes *all* the nodes in the top control point of the control stack.

`QApply` n T — parametrized the transform T with the top n elements of the classical stack and then applies the parametrized transform to quantum stack. Control is respected because of the labelling of the quantum stack.

The function $cTrans$ in the transition for `QApply` must first create the transform. In most cases, this is a fixed transform (e.g., Not, Hadamard), but both rotate and the UM transforms are parametrized. The transform rotate is used in the quantum Fourier transform and UM is the $a^x \mod N$ transform used in order finding.

When the top node is a qubit, the function expects its required number of qubits to be the top nodes. For example, a Hadamard expects only 1, a $swap$ expects 2 and an UM will expect as many qubits as N requires bits.

$$
\begin{aligned}
&(\text{AddCtrl}{:}\mathcal{C}, C, [(S_i, L(Q_i), D_i, N_i)]_{i=1,\cdots n})]) \\
&\qquad\qquad\qquad \Longrightarrow \quad (\mathcal{C}, \mathsf{id}{:}C, [(S_i, L(Q_i), D_i, N_i)]_{i=1,\cdots n})]) \\
&(\text{QCtrl}{:}\mathcal{C}, f{:}C, [(S_i, L(Q_i), D_i, N_i)]_{i=1,\cdots n})]) \\
&\qquad\qquad\qquad \Longrightarrow \quad (\mathcal{C}, (g \circ f){:}C, [(S_j', L(Q_j)', D_j')]_{j=1,\cdots m})]) \\
&(\text{UnCtrl}{:}\mathcal{C}, f{:}C, [(S_i, L(Q_i), D_i, N_i)]_{i=1,\cdots n})]) \\
&\qquad\qquad\qquad \Longrightarrow \quad (\mathcal{C}, C, [(S_j'', L(Q_j)'', D_j'')]_{j=1,\cdots p})]) \\
&(\text{QApply } m\ t{:}\mathcal{C}, (v_1{:}\cdots{:}v_m{:}S), L(Q), D, N) \\
&\qquad\qquad\qquad \Longrightarrow \quad (\mathcal{C}, S, \mathsf{cTrans}([v_1, \ldots, v_m], t, L(Q)), D, N)
\end{aligned}
$$

Figure 5.12: Transitions for unitary transforms

When a transform is applied to a datatype node the machine will attempt to rotate up the required number of qubits to the top, perform the operation and then re-rotate the datatype node back to the top.

The first step is to rotate the bound nodes of the datatype node starting at the left and proceeding to the right. Left to right is determined by the ordering in the original constructor expression used to create the datatype node. The machine will throw an exception if there are insufficient bound nodes (e.g., Nil for a list) or if the rotation would be indeterminate. The machine considers a rotation for a transform to be indeterminate whenever the subject datatype node has more than one sub-stack. For example, this means a transform can not be applied to an Either that is a mix of Left(|0>) and Right(|1>).

When the rotation of the qubits succeeds the function will transform the top parts of the stack into a matrix Q of appropriate size (2×2 for a $1-$qubit transform, 16×16 for a $4-$qubit and so forth) with entries in the matrix being the sub-stacks of the qubits used in the transform.

At this point, the control labelling of the quantum stack is considered and one of the following four transforms will happen. If the actual transform is named T, the result will be:

$$\text{cTrans } \mathsf{T}\ \mathsf{L}(\mathsf{Q}) = \begin{cases} \mathsf{L}(\mathsf{Q}) & \mathsf{L} = \texttt{IdOnly} \\ \mathsf{L}(\mathsf{TQ}) & \mathsf{L} = \texttt{Left} \\ \mathsf{L}(\mathsf{QT}^*) & \mathsf{L} = \texttt{Right} \\ \mathsf{L}(\mathsf{TQT}^*) & \mathsf{L} = \texttt{Full} \end{cases}$$

Following this the quantum stack is reformed from the resulting matrix.

5.4.9 Function calling and returning

The `Call` and `Return` instructions are used for function calling. The `Call` instruction is the only instruction that needs to directly work on QSM, the infinite list of CQSM items. The transition for this is defined in terms of a subordinate function *enterF* which is defined on BQSM. Its transition is also described below.

Recall the QS-machine stages have the states:

$$\mathrm{BQSM} = (\mathcal{C}, S, Q, D, N)$$

$$\mathrm{CQSM} = (\mathcal{C}, C, [(S, L(Q), D, N)])$$

$$\mathrm{QSM} = \mathrm{Inflist}(\mathcal{C}, C, [(S, L(Q), D, N)])$$

For the state QSM, an infinite list will be expressed as

$$H_0 \blacktriangleright T = H_0 \blacktriangleright H_1 \blacktriangleright H_2 \blacktriangleright \cdots$$

where H is an element of the correct type for the infinite list.

The instructions do the following tasks:

`Call` n lbl — Calls the subroutine at lbl, copying the top n elements of the classical stack to a classical stack for the subroutine.

`Return` n — Uses the return label in the head of the dump to return from the subroutine. It also copies the top n elements from the classical stack and places them on top of the saved classical stack from the dump element.

$$
\begin{array}{l}
(\mathrm{Call}\ n \triangleright\mathcal{C}_C{:}\mathcal{C}, C, [(S_i, L(Q)_i, D_i, N_i)]])_0 \blacktriangleright T \\
\qquad\qquad\qquad\qquad \Longrightarrow\ (\mathcal{C}, C, [(S_i, L(\emptyset)_i, D_i, N_i)]])_0 \blacktriangleright \mathit{lift}\ (\mathit{enterf}\ n \triangleright\mathcal{C}_C)\ T \\
\mathit{enterf}\ n \triangleright\mathcal{C}_C(\mathcal{C}, v_1{:}\cdots{:}v_n{:}S, Q, D, N) \\
\qquad\qquad\qquad\qquad \Longrightarrow\ (\mathcal{C}_C, [v_1, \ldots, v_n], Q, R(S, \triangleright\mathcal{C}){:}D, N) \\
(\mathrm{Return}\ n, v_1{:}\cdots{:}v_n{:}S', Q, R(S, \triangleright\mathcal{C}){:}D, N) \\
\qquad\qquad\qquad\qquad \Longrightarrow\ (\mathcal{C}, [v_1, \ldots, v_n]{:}S, Q, D, N)
\end{array}
$$

Figure 5.13: Transitions for function calls.

To illustrate how `Call` is being processed, consider the following diagram:

$$
\begin{array}{lrrrr}
(0) & M_0 \blacktriangleright & M_1 \blacktriangleright & M_2 \blacktriangleright & M_3 \blacktriangleright \ldots \\
(1)\ (\texttt{Call}\ f): & 0 \blacktriangleright & f\cdot M_1 \blacktriangleright & f\cdot M_2 \blacktriangleright & f\cdot M_3 \blacktriangleright \ldots \\
(2)\ (\texttt{Call}\ f): & 0 \blacktriangleright & 0 \blacktriangleright & f\cdot f\cdot M_2 \blacktriangleright & f\cdot f\cdot M_3 \blacktriangleright \ldots \\
(3)\ (\texttt{Call}\ f): & 0 \blacktriangleright & 0 \blacktriangleright & 0 \blacktriangleright & f\cdot f\cdot f\cdot M_3 \blacktriangleright \ldots \\
 & & \vdots & &
\end{array}
$$

At the start, in line (0), the machine has state $M_0 \blacktriangleright M_i$. After the first call to f, at line (1), the head of the infinite list state has been zeroed out, indicating divergence. However, at every position further down the infinite list, the subroutine f is entered.

Continuing to line (2) and calling f again, the divergence has moved one position to the right and we now have a state of $0 \blacktriangleright 0 \blacktriangleright f\cdot f\cdot M_i$. Line (3) follows the same pattern. Thus, the further along in the infinite list one goes, the greater the *call depth*.

The `Call` and `Return` instructions use a dump element as part of subroutine linkage. The `Call` i lbl instruction creates a dump element to store the current classical stack and the address of the instruction following the `Call` instruction. The `Return` n

instruction will use the top dump element to reset the code pointer to the saved return location. `Return` also takes the classical stack from the top dump element and the top n values from the current classical stack are added to the top of it. `Return` then removes the top dump element.

Chapter 6

Conclusions and Future Work

6.1 Conclusions

The language L-QPL, as presented here and in the appendices meets the goal of being a language usable for experimenting with quantum algorithms.

As seen in appendix C on page 145, L-QPL has the capability to implement various quantum algorithms such as quantum teleportation and Grover's search algorithm. Experimentation with the algorithms is possible, e.g., determining the appropriate number of times to apply the Grover transform for a given number of qubits, or determining the actual likelihood of getting the correct result from the algorithm.

This thesis has provided a complete description of the language, including an operational semantics in terms of a quantum stack and a less formal description of individual statements and operations in the language.

The quantum stack has been thoroughly described and been used to give an interpretation of quantum circuit diagrams. It has been compared and contrasted with the Q-RAM machine. All of its operations have been completely described via an operational semantics.

The thesis as it stands will be useful for researchers investigating quantum programming languages and for those experimenting with quantum algorithms.

6.2 Future work

I believe there are many avenues of exploration still left open with respect to L-QPL and its quantum stack machine, some of which I have described below.

6.2.1 Language extensions

The programming language L-QPL, while currently interesting and useful, could use additional features to make it a fully functional and powerful tool for developing quantum algorithms.

Refinements to the type system

In addition to the current **qdata** construction, a corresponding construction for classical data creation could be added. This declared classical data would be held in a classical node on the quantum stack, but could be moved back and forth to the classical stack.

A type *aliasing* declaration and potentially a *class* system to allow for closed types may also be useful.

Transform definition

Currently the language has a finite set of built-in transforms, with the rotation transforms being parametrized by integer values. However, one common feature of quantum algorithms seems to be the application of generic reversible classical computation on bits as a unitary transformation on qubits.

This is typically done by transforming the computation to a reversible bit computation and then extending it by linearity to a qubit computation.

I would like to add supporting syntax and semantics to the language to accomplish this, without sacrificing the compile time type safety.

Input/Output

The ability to allow a program to read in values and print out results, likely outside of the bounds of any qubit manipulation, is a necessity for future development.

Miscellaneous enhancements

Additional base types in the language, such as characters and floating point numbers, would be useful in some algorithms and in I/O.

6.2.2 Semantics

Creation of a categorical semantics for L-QPL, which matches the operational semantics, is a vital step in further understanding and exploration of the language.

Special attention will need to be given to higher-order functions, input / output and the area of quantum control.

6.2.3 Program transformation

The applicability of program transformation to quantum programming languages and quantum algorithms has not been explored yet. Starting with specifics in L-QPL, it will be interesting to explore general mechanisms for program transformation.

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Appendix A

BNF description of the Linear Quantum Programming Language

A.1 Program definition

L-QPL programs consist of a series of definitions at the global level. These are either *data definitions* which give a description of an algebraic data type *procedure definitions* which define executable code.

```
<Linearqplprogram>   :: <global definitions>

<global_definitions> :: <global_definitions> <global_definition>
     | empty

<global_definition> :: <data_definition>
     | <procedure_definition>
```

A.2 Data definition

A data definition consists of declaring a type name, with an optional list of type variables and a list of constructors for that type. It is a semantic error to have different types having the same constructor name, or to redeclare a type name.

Constructor definitions allow either fixed types or uses of the type variables mentioned in the type declaration.

```
<data_definition> :: <type_definition> '='
                     '{' <constructor_list> '}'

<type_definition> :: 'type' <constructorid> <id_list>

<constructor_list>:: <constructor> <more_constructor_list>
```

```
<more_constructor_list> ::
     '|' <constructor> <more_constructor_list>
     | {- empty -}

<constructor> :: <constructorid> '(' <typevar_list> ')'
     | <constructorid>

<typevar_list> :: <typevar> <moretypevar_list>

<moretypevar_list> :: ',' typevar moretypevar_list
     | {- empty -}

<typevar> :: <identifier>
     | <identifier>
     | <constructor>
     | <constructor> '(' <typevar_list> ')'
     | <builtintype>

<builtintype>:: 'Qubit'  | 'Int' | 'Bool'
```

A.3 Procedure definition

Procedures may only be defined at the global level in a L-QPL program. The definition consists of a procedure name, its input and output formal parameters and a body of code. Note that a procedure may have either no input, no outputs or neither.

The classical and quantum inputs are separated by a '|'. Definitions with either no parameters or no classical parameters are specific special cases.

```
<procedure_definition> :: <identifier> '::'
         '(' <parameter_definitions> '|'
             <parameter_definitions> ';'
             <parameter_definitions>  ')'
                 '='  <block>
     | <identifier> '::'
         '(' <parameter_definitions> ';'
             <parameter_definitions>  ')'
                 '='  <block>
     | <identifier> '::' '(' ')' '=' <block>

<parameter_definitions> :: <parameter_definition>
          <more_parameter_definitions>
     | {- empty -}

<more_parameter_definitions> :: ',' <parameter_definition>
```

```
          <more_parameter_definitions>
     | {- empty -}

<parameter_definition> :: <identifier> ':' <constructorid>
          '(' <typevar_list> ')'
     | <identifier> ':' <constructorid>
     | <identifier> ':' <builtintype>
```

A.4 Statements

Although L-QPL is a functional language, the language retains the concept of *statements* which provide an execution flow for the program.

The valid collections of statements are *blocks* which are lists of statements.

```
<block> :: '{' <stmtlist> '}'

<stmtlist>:: <stmtlist> ';' <stmt>
     | <stmtlist> ';'
     | <stmt>
     | {- empty -}
```

Statements are broadly grouped into a few classes.

A.4.1 Assignment

Variables are created by assigning to them. There is no ability to separately declare them. Type unification will determine the appropriate type for the variable.

```
<stmt> :: <identifier> '=' <exp>
```

A.4.2 Case statements

These are **measure**, **case**, **use**, **discard** and the classical assign, :=. These statements give the programmer the capability to specify different processing on the sub-stacks of a quantum variable. This is done with dependant statements. For **measure** and **case**, the dependent statements are in the block specified in the statement. For **use**, they may be specified explicitly, or they may be all the statements following the **use** to the end of the enclosing block. The classical assign is syntactic sugar for an assignment followed by a **use** with no explicit dependent statements.

The **discard** statement is grouped here due to the quantum effects. Doing a discard of a qubit is equivalent to measuring the qubit and ignoring the results. This same pattern is followed for discarding quantum variables of all types.

```
(<stmt> continued)
     | 'case' <exp> 'of' <cases>
     | 'measure' <exp> 'of' <zeroalt> <onealt>
     | <identifier> ':=' <exp>
     | 'use' <identifier_list> <block>
     | 'use' <identifier_list>
```

A.4.3 Functions

This category includes procedures and transforms. A variety of calling syntax is available, however, there are no semantic differences between them.

```
(<stmt> continued)
     | '(' <identifier_list> ') '='
           <callable>  '(' <exp_list> ')'
     | '(' <identifier_list> ') '='
           <callable>  '(' <exp_list> '|' <exp_list> ')'
     | <callable> <ids>
     | <callable> '(' <exp_list> ')' <ids>
     | <callable> '(' <exp_list> '|' <exp_list> ';'
           <ids> ')'
```

A.4.4 Blocks

The block statement allows grouping of a series of statements by enclosing them with { and }. An empty statement is also valid.

```
(<stmt> continued)
     |  <block>
     | {- empty -}
```

A.4.5 Control

L-QPL provides a statement for classical control and one for quantum control. Note that quantum control affects only the semantics of any transformations applied within the control. Classical control requires the expressions in its guards (see below) to be classical and not quantum.

```
(<stmt> continued)
     | 'if' guards
     | <stmt> '<=' <control_list>
```

A.4.6 Divergence

This signifies that this portion of the program does not terminate. Statements after this will have no effect.

```
(<stmt> continued)
     | 'zero'
```

A.5 Parts of statements

The portions of statements are explained below. First is *callable* which can be either a procedure name or a particular unitary transformation.

```
<callable> :: <identifer> | <transform>
```

The alternatives of a measure statement consist of choice indicators for the base of the measure followed by a block of statements.

```
<zeroalt> ::  '|0>' '=>' <block>

<onealt> :: '|1>' '=>' <block>
```

The if statement requires a list of *guards* following it. Each guard is composed of a classical expression that will evaluate to true or false, followed by a block of guarded statements. The statements guarded by the expression will be executed only when the expression in the guard is true. The list of guards must end with a default guard called **else**. Semantically, this is equivalent to putting a guard of true.

```
<guards> :: <freeguards> <owguard>

<freeguards> :: <freeguard> <freeguards>
     | {- empty -}

<freeguard> :: <exp> '=>' <block>

<owguard> :: 'else' '=>' <block>
```

When deconstructing a data type with a **case** statement, a pattern match is used to determine which set of dependent statements are executed. The patterns allow the programmer to either throw away the data element (using the '_' special pattern), or assign it to a new identifier.

```
<cases> :: <case> <more_cases>

<more_cases> :: {- empty -}
     | <case> <more_cases>

<case> :: <caseclause> '=>' <block>

<caseclause> ::  <constructorid> '(' <pattern_list> ')'
     | <constructorid>

<pattern_list>:: <pattern> <more_patterns>

<more_patterns> :: ',' <pattern> <more_patterns>
     | {- empty -}

<pattern> :: <identifier> | '_'
```

A.6 Expressions

L-QPL provides standard expressions, with the restriction that arithmetic expressions may be done only on classical values. That is, they must be on the classical stack or a constant.

The results of comparisons are Boolean values that will be held on the classical stack.

```
<exp>:: <exp0>

<exp0>:: <exp0> <or_op> <exp1> | <exp1>

<exp1>:: <exp1> '&&' <exp2> | <exp2>

<exp2>:: '~' <exp2> | <exp3>  | <exp3> <compare_op> <exp3>

<exp3>:: <exp3> <add_op> <exp4> | <exp4>

<exp4>:: <exp4> <mul_op> <exp5> | <exp5>

<exp5>:: <exp5> <shift_op> <exp6> | <exp6>

<exp6>:: <identifier> | <number> | 'true' | 'false'
     | '(' <exp> ')'
     | <constructorid> '(' <exp_list> ')'
     | <constructorid>
     | <identifier> '('  ')'
     | <identifier> '(' <exp_list> ')'
     | <identifier> '(' <exp_list> ';' ids ')'
     | '|0>' | '|1>'

<exp_list>:: <exp> <more_exp_list>

<more_exp_list>:: ',' <exp> <more_exp_list>
     | {- empty -}
```

A.7 Miscellaneous and lexical

These are the basic elements of the language as used above. Many of these items are differentiated at the lexing stage of the compiler.

```
<idlist> :: <identifier> more_ids
     | {- empty -}

<more_ids> :: <identifier> more_ids
     | {- empty -}

<control_list>:: <control> <more_control_list>
```

```
<more_control_list>:: ',' <control> <more_control_list>
     | {- empty -}

<control>:: <identifer>   | '~' <identifer>

<opt_identifier_list>:: <identifier_list>
     | {- empty -}

<identifier_list>:: <identifier> <more_idlist>

<more_idlist>:: ',' <identifier> <more_idlist>
     | {- empty -}

<or_op>:: '||' | '^'

<compare_op>:: '==' | '<' | '>' | '=<' | '>=' | '=/='

<add_op>::'+' | '-'

<mul_op>::'*' | 'div'  | 'rem'

<shift_op>::'>>' | '<<'

<transform>:: <gate>
     | <transform> *o* transform

<gate> :: 'Had' | 'T'  | 'Phase' | 'Not' |  'RhoX'
     | 'Swap' | 'Rot '| 'RhoY ' | '  RhoZ ' | 'Inv-'<gate>

<identifier> :: <lower> | <identifier><letterOrDigit>
<constructorid :: <upper> | <constructorid><letterOrDigit>
<letterOrDigit> :: <upper>|<lower>|<digit>
<number> :: ['+'|'-'] <digit>+
<lower> ::   'a' | 'b' | 'c' | 'd' | 'e' | 'f' | 'g'
     | 'h' | 'i' | 'j' | 'k' | 'l' | 'm' | 'n' | 'o'
     | 'p' | 'q' | 'r' | 's' | 't' | 'u' | 'v' | 'w'
     | 'x' | 'y' | 'z'
<upper> ::   'A' | 'B' | 'C' | 'D' | 'E' | 'F' | 'G'
     | 'H' | 'I' | 'J' | 'K' | 'L' | 'M' | 'N' | 'O'
     | 'P' | 'Q' | 'R' | 'S' | 'T' | 'U' | 'V' | 'W'
     | 'X' | 'Y' | 'Z'
<digit> ::   '0' | '1' | '2' | '3' | '4' | '5' | '6'
     | '7' | '8' | '9'
```

Appendix B

Quantum stack machine additional details

B.1 Instructions

Creation of reasonable set of instructions, balancing brevity and usefulness has been an interesting task. The list of instructions and brief descriptions of them are presented in table B.1. The transitions of these are presented formally in section 5.4 on page 96.

Table B.1: QSM instruction list

Instruction	**Arguments**	**Description**
`QLoad`	*nm:Name, k::qubit*	Creates qubit named *nm* and sets the value to $\vert\mathtt{k}\rangle$.
`QMove`	*nm:Name*	Creates an integer or Boolean named *nm* and sets its value to the top of the classical stack.
`QCons`	*nm:Name, c::constructor*	Creates a data type element with the value *c*. Note that if the constructor requires sub-elements, this will need to be followed by `QBind` instructions.
`QBind`	*nm:Name*	Binds the node $[\mathtt{nm}]$ to the data element currently on the top of the stack.
`QDelete`	$\emptyset$	Deletes the top node and any bound nodes in the quantum stack.
`QDiscard`	$\emptyset$	Discards the node on top of the quantum stack.
`QUnbind`	*nm:Name*	Unbinds the first bound node from the data element at the top of the stack and assigns it as *nm*.
`QPullup`	*nm::name*	Pulls the node named *nm* to the top of the quantum stack.

Continued on next page

Instruction	Arguments	Description
`QName`	*nm1::name, nm2::name*	Renames the node named *nm1* to *nm2*.
`AddCtrl`	$\emptyset$	Marks the start of a control point in the control stack. Any following `QCtrl` instructions will add the top node to this control point.
`QCtrl`	$\emptyset$	Moves the top element of the quantum stack to the control stack. Recursively moves any bound nodes to the control stack when the element is a constructed data type.
`UnCtrl`	$\emptyset$	Moves all items in the control stack at the current control point back to the quantum stack.
`QApply`	*i::Int,t:Transform*	Parametrizes the transform T with the top i elements of the classical stack and applies it to the quantum stack.
`Measure`	*l0::Label, l1::Label*	Measures the qubit on top of the quantum stack and sets up the dump for execution of the code at *l0* for the 00 sub-branch and the code at *l1* for the 11 sub-branch.
`Split`	*cls::[(constructor, Label)]*	Splits the data node at the top of the quantum stack and sets up the dump for execution of the code at the i-th label for the i-th sub-branch.
`Use`	*lbl::Label*	Uses the classical (integer or Boolean) node on top of the quantum stack and sets up the dump to for execution of the code at *lbl* for each of the sub-branches.
`EndQC`	$\emptyset$	Merges the current quantum stack with the results stack of the dump, activates the next partial stack to be processed and jumps to the code at the corresponding label. When there are no more partial stacks, the instruction merges the current stack with the the results stack and sets that as the new quantum stack.

Continued on next page

Instruction	**Arguments**	**Description**
`Call`	*i::Int*, *ep::EntryPoint*	For the first element of the infinite list of states, sets the values at the leaves of the quantum stack to 0. For the remainder of the list, the instruction jumps to the subroutine at *ep*, saving the return location and classical stack on the dump. It copies the top *i* elements of the classical stack for the subroutine.
`Return`	*i::Int*	Restores the location and classical stack from the dump, copies the top *i* items of the current classical stack to the top of the restored classical stack.
`Jump`	*lbl::Label*	Jumps *forward* to the label *lbl*.
`CondJump`	*lbl::Label*	If the top of the classical stack is the value `false`, jumps *forward* to the label *lbl*.
`NoOp`	$\emptyset$	Does nothing.
`CGet`	*i::Int*	Copies the *i*-th element of the classical stack to the top of the classical stack. A negative value for *i* indicates the instruction should copy the $\lvert \mathtt{i} \rvert^{\text{th}}$ value from the bottom of the classical stack.
`CPut`	*i::Int*	Copies the top of the classical stack to the *i*-th element of the classical stack. A negative value for *i* indicates the instruction should place the value into the $\lvert \mathtt{i} \rvert^{\text{th}}$ location from the bottom of the classical stack.
`CPop`	$\emptyset$	Pops off (and discards) the top element of the classical stack.
`CLoad`	*v::Either Int Bool*	Pushes *v* onto the classical stack.
`CApply`	*op::Classical Op*	Applies *op* to the top elements of the classical stack, replacing them with the result of the operation.

B.2 Translation of L-QPL to stack machine code

This section will discuss the code produced by the various statements and expressions in an L-QPL program. An L-QPL program consists of a collection of data definitions and procedures. Data definitions do not generate any direct code but do affect the code generation of statements and expressions.

Each procedure will generate code. A procedure consists of a collection of statements each of which will generate code. Some statements may have other statements of expressions as dependent pieces, which again will generate code.

The code generation in the compiler, and the description here, follows a standard recursive descent method.

B.2.1 Code generation of procedures

The code generated for each procedure follows a standard pattern of: procedure entry; procedure statements; procedure exit. The procedure statements portion is the code generated for the list of statements of the procedure, each of which is detailed in appendix B.2.2 on the following page.

Procedure entry

Each procedure is identified in QSM by an entry point, using an assembler directive. This directive is a mangled name of the procedure, followed by the keyword `Start`. The only exception to this is the special procedure `main` which is generated without mangling. `main` is always the starting entry point for a QSM program.

Procedure exit

The end of all procedures is denoted by another assembler directive, `EndProc`. For all procedures except `main`, the code generation determines how many classical variables are being returned by the procedure and emits a `Return` n instruction, where n is that count[1].

Procedure body

The code for each statement in the list of statements is generated and used as the body of the procedure. As an example, see the coin flip code and the corresponding

```
qdata Coin = {Heads | Tails}
cflip ::( ; c:Coin) =
{   q = |0>;
    Had q;
    measure q of
       |0> => {c = Heads}
       |1> => {c = Tails}
}
main :: () =
{   c = cflip()}
```

(a) Coin flip code

```
CFlip_fcdlbl0 Start
    QLoad q |0>
    QApply 0 !Had
    Measure l0 l1
    Jump l3
l0  QDiscard
    QCons b #False
    EndQC
l1  QDiscard
    QCons b #True
    EndQC
l3  QPullup b
    Return 0
    EndProc

main Start
    Call 0 CFlip_fcdlbl0
    NoOp
    EndProc
```

(b) Generated code

Figure B.1: L-QPL and QSM coin flip programs

generated QSM code in figure B.1.

B.2.2 Code generation of statements

Each statement in L-QPL generates code. The details of the code generation for each statement are given in the following pages, together with examples of actual generated code.

[1]This functionality is currently not available in L-QPL, but may be re-introduced at a later date.

Assignment statements

The sub-section describes code generation for quantum assignment statements and assignments to variables on the classical stack. The classical assignment (:=) statement is described with the **use** statement below, as it is syntactic sugar for that statement.

An assignment of the form `i` = ⟨`expr`⟩ is actually broken down into 5 special cases. The first is when the left hand side is an in-scope variable that is on the classical stack. The other four all deal with the case of a quantum variable, which is either introduced or overwritten. The four cases depend on the type of expression on the right hand side. Each paragraph below will identify which case is being considered and then describe the code generation for that case.

Left hand side is a classical variable. In this case, generate the code for the expression on the right hand side (which will be classical in nature). This leaves the expression value at the top of the classical stack. Now, emit a `CPut` instruction which will copy that value into the location of the classical variable.

```
i = 5;        ⟹        CLoad  5
                       CPut  −2
```

Right hand side is a classical expression. First, generate the expression code, which leaves the value on the top of the classical stack. Then emit a `QMove` instruction with the name of the left hand side. This will create a new classical node, which will be set to the value of the top of the classical stack.

```
i = 5;        ⟹        CLoad  5
                       QMove  i
```

Right hand side is a constant qubit. Emit the `QLoad` instruction with the qubit value and the left hand side variable name.

```
q = |1>;          ⟹        QLoad q |1>
```

Right hand side is an expression call. First, emit the code for the expression call. This will leave the result quantum value on the top of the quantum stack. If the formal name given by the procedure definition is the same as the left hand side name, do nothing else, as the variable is already created with the proper name. If not, emit a `QName` instruction to rename the last formal parameter name to the left hand side name.

```
random :: (maxval :Int;              CLoad 15
        rand :Int) =          ⟹     QMove c18
  {...}                              QName c18 maxval //In
...                                  Call 0 random_fcdlbl0
x = random(15);                      QName rand x //Out
```

Right hand side is some other expression. Generate the code for the expression. Check the name on the top of the stack. If it is the same as the left hand side name, do nothing else, otherwise emit a `QName` instruction.

```
outqs = Cons(q, inqs');            QCons c4 #Cons
                          ⟹       QBind inqs'
                                   QBind q
                                   QName c4 outqs
```

Measurement code generation

Measurement will always have two subordinate sets of statements, respectively for the $|0\rangle$ and $|1\rangle$ cases. The generation for the actual statement will handle the requisite branching.

The code generation first acquires three new labels, m_0, m_1 and m_f. It will then emit a `Measure` m_0 m_1 statement, followed by a `Jump` m_f. Recall from the transitions in sub-section 5.4.1 on page 96 that when the machine executes the `Measure` instruction, *it then generates and executes a* `EndQC` instruction. The `Jump` will be executed when all branches of the qubit have been executed.

Then, for each of the two sub blocks ($i \in \{0, 1\}$), I emit a `Discard` labelled with m_i. This is followed by the code generated from the corresponding block of statements. Finally a `EndQC` is emitted.

The last instruction generated is a `NoOp` which is labelled with m_f.

```
measure q of
  |0> => {n1 = 0}
  |1> => {n1 = 1};
```

$\Longrightarrow$

```
     QPullup  q
     Measure  l7  l8
     Jump  l9
l7   QDiscard
     CLoad  0
     QMove  n1
     EndQC
l8   QDiscard
     CLoad  1
     QMove  n1
     EndQC
l9   NoOp
```

Case statement code generation

Case statement generation is conceptually similar to that of measurement. The differences are primarily due to the variable number of case clauses and the need to instantiate the variables of the patterns on the case clauses.

As before, the expression will have its code generated first. Then, the compiler will use a function to return a list of triples of a constructor, its generated label and the corresponding code for each of the case clauses. The code generation done by that function is detailed below.

At this point, the code generation resembles measurement generation. The compiler generates a label c_f and emits a `Split` with a list of constructor / code label pairs which have been returned by the case clause generation. This is followed by emitting a `Jump` c_f. After the `Jump` has been emitted, the code generated by the case clause generation is emitted.

The final instruction generated is a `NoOp` which is labelled with c_f.

Case clause code generation. The code generation for a case clause has a rather complex prologue which ensures the assignment of any bound variables to the patterns in the clause.

First, the code generator gets a new label c_l and then calculates the unbinding code as follows: For each *don't care* pattern in the clause, code is generated to delete the corresponding bound node. This is done by first getting a new stack name nm, then adding the instructions `QUnbind` nm, `Pullup` nm and `QDelete` nm. This accomplishes the unbinding of that node and removes it from the quantum stack. Note that `QDelete` is used here to ensure that all subordinate nodes are removed and that no spurious data is added to the classical stack in the case of the don't care node being a classical value.

For each named pattern p, only the instruction `QUnbind` p is added.

The program now has a list of instructions that will accomplish unbinding of the variables. Note this list may be empty, e.g., the Nil constructor for List.

The clause generation now creates its own return list. When the unbinding list is not empty, these instructions are added first, with the first one of them being labelled with c_l.This is followed by a `Discard` which discards the decomposed data node. When the unbind list is empty, the first instruction is the `Discard` labelled by c_l.

Then, the code generated by the statements in the case clause block are added to the list. Finally, a `EndQC` is added at the end.

The lists from all the case clauses are combined and this is returned to the case code generation.

The example at the end of the use clause will illustrate both the case clause and the use statement code generation.

Use and classical assign code generation

As described earlier in sub-section 4.3.5 on page 75, the classical assignment, `v :=exp` is syntactic sugar for a variable assignment followed by a **use** statement. The code generation for each is handled the same way.

There are two different cases to consider when generating this code. The **use** statement may or may not have subordinate statements. In the case where it does not have any subordinate statements, (a classical assign or a use with no block), the scope of the classical variables in the use extends to the end of the enclosing block.

The case of a use with a subordinate block is presented first.

Use with subordinate block code generation. While similar to the generation for the measure and case statements, there are differences. The two main differences are that there is only one subordinate body of statements and that multiple variables may be used.

In the case where there is a single use variable nu, the generator gets the body label u_b and end label u_e. It then emits a `Pullup` nu, a `Use` u_b and a `Jump` u_e.

This is followed by emitting a `Discard` labelled by u_b and the code generated by the subordinate body of statements and a `EndQC` instruction. This is terminated by a `NoOp` labelled by u_e.

When there are multiple names $n_1, n_2, \ldots, n_j$, the generator first recursively generates code assuming the same body of statements but with a use statement that only has the variables $n_2, \ldots, n_j$. This is then used as the body of code for a use statement with only one variable n_1, which is generated in the same manner as in the above paragraph.

Use with no subordinate block. To properly generate the code for this, including the `EndQC` and end label, the generator uses the concept of delayed code. The prologue (`Use`, `Jump` and `Discard`) and epilogue (`EndQC`, `NoOp`) are created in the same manner as the use with the subordinate block. The prologue is emitted at the time of its generation. The epilogue, however, is added to a push-down stack of delayed code which is emitted at the end of a block. See the description of the block code generation for more details on this. These items are illustrated in figure B.2 on the following page.

Conditional statements

The **if** ... **else** statement allows the programmer to specify an unlimited number of classical expressions to control blocks of code. Typically, this is done within a **use** statement based upon the variables used.

The statement code generation is done by first requesting a new label, g_e. This label is used in the next step, generating the code for all of the guard clauses. The generation is completed by emitting a `NoOp` instruction labelled by g_e.

Guard clauses. Each guard clause consists of an expression and list of statements. The code generator first emits the code to evaluate the expression. Then, a new label

```
case l of
  Nil => {i = 0}
  Cons (_, l1) => {
    n= len(l1);
    use n;
    i = 1 + n;
}
```

$\Longrightarrow$

```
   Split (#Nil,l0) (#Cons,l1)
       Jump l4
l0     QDiscard
       CLoad 0
       QMove i
       EndQC
l1     QUnbind c0
       QUnbind l1
       QDiscard
       QPullup c0
       QDiscard
       QPullup l1
       QName l1 l
       Call 0 len_fcdl0
       QName i n
       QPullup n
       Use l2
       Jump l3
l2     QDiscard
       CLoad 1
       CGet 0
       CApply +
       QMove i
       EndQC //For Use
l3     NoOp
        EndQC //For Split
l4     NoOp
```

Figure B.2: QSM code generated for a case and use statement.

g_l is requested and the instruction `CondJump` g_l is emitted. The subordinate statements are generated and emitted, considering them as a single block. The concluding instruction is a `Jump` g_e.

At this point, if there are no more guard clauses, a `NoOp` instruction, labelled by g_l is emitted, otherwise the code generated by the remaining guard clauses is labelled by g_l and emitted.

```
if b == 0 => {
   theGcd = a;
} else => {
   (theGcd) =
      gcd(b, a mod b);
}
```

$\Longrightarrow$

```
     CGet -2
     CLoad 0
     CApply ==
     CondJump lbl2
     CGet -1
     QMove theGcd
     Jump lbl1
lbl2 CLoad True //else
     CondJump lbl3
     CGet -1
     CGet -2
     CApply %
     QMove c0
     CGet -2
     QMove c1
     QName c1 a
     QName c0 b
     Call 0 gcd_fcdlbl0
     Jump lbl1
lbl3 NoOp
lbl1 NoOp
```

Function calling and unitary transforms

The code generation of these two statements is practically the same, with the only difference being that built in transformations have a special instruction in QSM, while executing a defined function requires the `Call` instruction.

In each case, the statement allows for input classical and quantum expressions and output quantum identifiers.

The first step is the generation of the code for the input classical expressions. These are generated in reverse order so that the first parameter is on the top of the classical stack, the second is next and so forth. Then, the input quantum expressions are generated, with names of these expressions being saved. Note that it is possible to use expressions which are innately classical (e.g., constants and variables on the classical stack) as a quantum expression. The compiler will generate the code needed to lift it to a quantum expression. See appendix B.2.4 on page 144 for the details.

At this stage, the two types differ slightly. For the unitary transformation case, the code "`QApply` n !t" is emitted, where n is the number of classical arguments and t is the name of the built in transform. The exclamation mark is part of the QSM assembler syntax for transform names. In a defined function, renames of the input quantum expressions to the names of the input formal ids are generated, by emitting a series of `QName` instructions. This is followed by emitting a `Call` n f, where n is again the number of classical arguments and f is the internally generated name of the function.

In both cases, the code checks the formal return parameter names against the list of return value names. For each one that is different, a `QName` $frml$ $retnm$ is emitted.

See the previous code list between the `CondJump lbl3` and `lbl3 NoOp` for an example of this.

B.2.3 Code generation of expressions

In the previous sub-section, a number of examples of code generation were given. These also illustrated most of the different aspects of expression code generation. A few additional examples are given below.

Generation of constants

There are three possible types of constants, a Boolean, an integer or a qubit. Note that constructors are considered a different class of expression.

For both Boolean and integers, the compiler emits a `CLoad` val instruction. For a qubit, it creates a new name q and emits a `QLoad` q qbv instruction.

Examples of these may be seen in the sub-sub-section on assignments.

Generation of classical arithmetic and Boolean expressions

In all cases, these types of expressions are calculated solely on the classical stack. Whenever the generator encounters an expression of the form

$$e_1 \text{ op } e_2$$

it first emits the code to generate e_2, followed by the code to generate e_1. This will leave the result of e_1 on top of the stack with e_2's value right below it. It then emits the instruction `CApply` op, which will apply the operation to the two top elements, replacing them with the result.

The Boolean *not* operation is the only operation of arity 1. Code generation is done in the same manner. The generator emits code for the expression first, followed by `CApply` $\neg$.

See under Guard clauses, appendix B.2.2 on page 137 for examples.

Generation of variables

The semantic analysis of the program will split this into two cases; classical variables and quantum variables. Each are handled differently.

Classical variables. These variables are on the classical stack at a specific offset. The use of them in an expression means that they are to be *copied* to the top of the classical stack. The code emitted is `CGet offset`, where `offset` is the offset of the variable in the classical stack.

Quantum variables. In this case, the variables are at a specific address of the quantum stack. These variables are not allowed to be copied, so the effect of this code is to rotate the quantum stack until the desired variable is at the top. The other consideration is that these variables have a linearity implicitly defined in their usage. In the compiler, this is handled by the semantic analysis phase, but the code generation needs to also consider this. The compiler will add this variable to a *delayed deletion* list. After completion of the statement with this variable expression, the variable will be deleted unless:

- The statement has deleted it. (Measure of a qubit, for example, will directly generate the deletion code.)
- It is recreated as the result of an assignment, function call or transformation.

Code generation for expression calls

Each expression call is generated in substantially the same way as the code for a call statement as in appendix B.2.2. The only difference is that the name of the final return variable will not be known and is therefore set to the same name as the name of the last output formal parameter. As an example, consider:

```
gcd::(a:Int, b:Int; theGcd:Int)=
{ use a,b in
  { if b == 0 => { theGcd   = a}
    else      => { (theGcd) = gcd(b, a mod b)}
  }
}
```

Suppose this is defined in a program, and at some point, it is called as an expression in the program : gcd(5,n). The code generated for this expression will then leave the integer node named *theGcd* on the top of the quantum stack.

Generation of constructor expressions

These expressions are used to create new data type nodes, such as lists, trees etc. Constructors are similar to functions in that they expect an expression list as input and will return a new quantum variable of a specific type. In L-QPL they are somewhat simpler as the input expressions are all expected to be quantum and there is a single input only. Just as in function calls, any classical expressions input to the constructor will be lifted to a quantum expression.

The first step is for the compiler to emit code that will evaluate and lift any of the input expressions. The names of each of these expressions is saved. Then, it creates a new name $\mathtt{d_c}$ and emits the code `QCons` $\mathtt{d_c}$ `#cid`.

The final stage is to emit a `QBind` $\mathtt{nm}_{e_i}$ for each of the input expression, *in reverse order* to what was input. The next example illustrates this.

```
lt = Cons(|0>,
        Cons(|0>,
        iTZQList(2×n)));
```

$\Longrightarrow$

```
    QLoad c22 |0>
    QLoad c23 |0>
    CLoad 2 //for the call
    CGet −1
    CApply ×
    QMove c24
    QName c24 n
    Call 0 iTZQList_fcdlbl5
    QCons c25 #Cons
    QBind nq
    QBind c23 //c23:nq
    QCons c26 #Cons
    QBind c25
    QBind c22 //c22:c23:nq
    QName c26 lt
```

B.2.4 Lifting of classical expressions to quantum expressions.

When the compiler requires a quantum expression, but has been given a classical one, it first generates the classical expression. This leaves the expression value on top of the classical stack. The compiler will now generate a new unique name l_c and emit the instruction `QMove` l_c. This now moves the value from the classical stack to the quantum stack.

Appendix C

Example L-QPL programs

C.1 Basic examples

This section covers a number of the standard examples of quantum algorithms covered in most introductions to the subject.

C.1.1 Quantum teleportation function

The L-QPL program shown in figure C.1 is an implementation of a function that will accomplish quantum teleportation as per the circuit shown previously in figure 2.9 on page 22. (See also [Wat] or [NC00]). It also provides a separate function to place two qubits into the EPR state.

Note that the teleport function, similarly to the circuit, does not check the precondition that qubits a and b are in the EPR state, which is required to actually have teleportation work.

```
prepare::( ; a:Qubit, b:Qubit)=
{ a = |0>;  b = |0>;
  Had a;
  Not b  <=  a;
}
teleport::(n:Qubit, a:Qubit, b:Qubit ; b:Qubit) =
{ Not a  <=  n ;
  Had n;
  measure a of
       |0> => {}   |1> => {Not b};
  measure n of
       |0> => {}   |1> => {RhoZ b}
}
```

Figure C.1: L-QPL code for a teleport routine

C.1.2 Quantum Fourier transform

The L-QPL program to implement the quantum Fourier transform in figure C.2 uses two recursive routines, qft and rotate. These functions assume the qubits to transform are in a List. (See also [Sel04] and either [Wat] or [NC00]).

The routine qft first applies the Hadamard transform to the qubit at the head of the list, then uses the rotate routine to recursively apply the correct Rot transforms controlled by the other qubits in the list. qft then recursively calls itself on the remaining qubits in the list.

```
#Import Prelude.qpl
rotate::(n:Int | h:Qubit, qbsIn :List(Qubit);
                 h:Qubit, qbsOut:List(Qubit))=
{ case qbsIn of
     Nil           => {qbsOut = Nil }
     Cons(hd, tl) =>
       { Rot(n) h     <= hd;
         rotate(n+1) h tl;
         qbsOut = Cons(hd,tl) }
}
qft::(qsIn:List(Qubit); qsOut:List(Qubit)) =
{ case qsIn of
    Nil           => {l = Nil}
    Cons(hd,tl) =>
      {  Had hd;
         rotate(2) hd tl;
         qft tl;
         qsOut = Cons(hd,tl) }
}
```

Figure C.2: L-QPL code for a quantum Fourier transform

C.1.3 Deutsch-Jozsa algorithm

The L-QPL program to implement the Deutsch-Jozsa algorithm is in figure C.3 on the following page with supporting routines in figure C.5, figure C.4, figure C.6. The hadList function is defined in figure C.31 on page 170.

The algorithm decides if a function is balanced or constant on n bits. This implementation requires supplying the number of bits / qubits used by the function, so that the input can be prepared. Additionally, it currently requires hand-writing the *oracle* for this function, shown in figure C.7. The oracle in this example is for a balanced function. (See also [Wat] or [NC00]).

The function dj creates an input list for the function, applies the Hadamard transform to all the elements of that list and then applies the oracle. When that is completed, the initial segment of the list is transformed again by Hadamard and then measured.

```
#Import initList.qpl
#Import prependnzeros.qpl
#Import hadList.qpl
#Import measureInps.qpl
#Import djoracle.qpl

dj::(size:Int |; resultType:Ftype)=
{  inlist = prependNZeroqbs(size | |1>);
   hadList inlist;
   djoracle inlist;
   inputs = initialList(inlist);
   hadList inputs;
   resultType = measureInputs(inputs);
}
```

Figure C.3: L-QPL code for the Deutsch-Jozsa algorithm

The function prependNZeroqbs creates a list of qubits when given a length and the last value. Assuming the parameters passed to the function were 3 and $|1\rangle$, this would return the list:

$$[|0\rangle, |0\rangle, |0\rangle, |1\rangle]$$

The initList function removes the last element of a list.

The measureInputs function recursively measures the qubits in a list. If any of them

```
#Import Prelude.qpl
prependNZeroqbs::(size:Int | last:Qubit;
                              resultList:List(Qubit))=
{     if (size == 0) => { resultList = Cons(last,Nil)}
      else           => { last'      = addNZeroqbs(size - 1 | last);
                          resultList = Cons(|0>,last') }
}
```

Figure C.4: L-QPL code to prepend n $|0\rangle$'s to a qubit

```
#Import Prelude.qpl
initialList::(inlist:List(a) ; outlist:List(a)) =
{    case inlist of
       Nil            =>
              {outlist = Nil}
       Cons(hd,tail) =>
              {outlist = initial(hd, tail)}
}
initial::(head:a, inlist:List(a) ;
                  outlist:List(a))=
{   init    = initialList(inlist);
    outlist = Cons(head,init)
}
```

Figure C.5: L-QPL code accessing initial part of list

measure to 1, it returns the value Balanced. If all of them measure to 0, it returns the value Constant.

```
#Import Prelude.qpl
qdata Ftype = {Balanced | Constant}
measureInputs::(inputs:List(Qubit) ; result:Ftype) =
{   case inputs of
      Nil           => {result = Constant} //All were zero
      Cons(hd, tail) =>
        {   measure hd of
              |0> => { result = measureInputs(tail)}
              |1> => { result = Balanced} }
}
```

Figure C.6: L-QPL code to measure a list of qubits

The oracle for the Deutsch-Jozsa algorithm is normally assumed to be a given. An actual implementation such as this requires an actual function to be provided. The function djoracle effects a transformation on the qubits in the input list by delegating to the function bal.

```
balanced :: (ctlqb:Qubit, inlist :List(Qubit) ;
             ctlqb:Qubit, outlist:List(Qubit))=
{ case inlist of
     Nil         => {outlist = Nil}
     Cons(hd,tl) =>
       { case tl of
           Nil => { Not hd    <= ctlqb;
                    outlist  = Cons(hd,Nil)}
           Cons(h,t) =>
             { balanced(ctlqb, t ; ctlqb,outt);
               outlist = Cons(hd,Cons(h,outt)) } }
}
djoracle::(inl:List(Qubit); outl:List(Qubit))=
{   case inl of
     Nil         => { outl = Nil}
     Cons(hd,tl)=> { balanced hd tl;
                     outl = Cons(hd,tl)}
}
```

Figure C.7: L-QPL code for Deutsch-Jozsa oracle

C.2 Hidden subgroup algorithms

The two algorithms presented in this section, the Grover search and Simons, are examples of the hidden subgroup problem.

C.2.1 Grover search algorithm

The L-QPL program to implement the Grover search algorithm is in figure C.8 on the following page with supporting routines in figure C.10, figure C.9, figure C.11. The hadList function is defined in figure C.31 on page 170 and the oracle for this implementation in figure C.12 on page 153.

For a specific function $f : \mathtt{bit}^n \to \mathtt{bit}$, the algorithm probabalistically determines the solution to $f(x) = 1$. Classically, this would require 2^n applications of f. The quantum algorithm requires $\mathcal{O}(\sqrt{2^n})$ applications. For the algorithm, first define

$$U_f \left|x\right\rangle = (-1)^{f(x)} \left|x\right\rangle \text{ and } U_0 \left|x\right\rangle = \begin{cases} \left|x\right\rangle & \text{if any } x \neq 0 \\ -\left|x\right\rangle & \text{if } x = 0^n \end{cases}$$

then:

- Start with n zeroed qubits and apply Hadamard to them.
- Apply $G = -H^{\otimes n} U_0 H^{\otimes n} U_f$ approximately $\sqrt{2^n}$ times.
- Measure the qubits, forming an integer and check the result.

In the implementation that follows, U_0 is given by the function phase, U_f is the function oracle and G is given by gtrans.

For a complete description and analysis of the algorithm, see [Wat] or [NC00].

The function **main** creates a zeroed qubit list for the function, applies the Hadamard transform to all the elements of that list and then applies the G transformation 4 times as this example is for a 4-bit function.

```
#Import intListConversion.qpl
#Import gtrans.qpl
main::()=
{ dataqbs = intToZeroQubitList(15 | );
  hadList dataqbs;
  doNGrovers(4) dataqbs;
  i = qubitListToInt(dataqbs);
}
```

Figure C.8: L-QPL code to call the grover search algorithm

The function intToZeroQubitList creates a list of zeroed qubits as long as the standard binary representation of the input number. For example, if it were passed the value 3, it would return a list of length 2. If it were passed the value 21 it would return a list of length 5.

The function qubitListToInt creates a probabilistic integer based on the values of the qubits in the list. Note that the list is assumed to be least significant digit first.

```
#Import Prelude.qpl
intToZeroQubitList::(n:Int | ; nq:List(Qubit))=
{ if  n == 0 => { nq  = Nil }
  else       => { nq' = intToZeroQubitList(n ≫ 1 |);
                  nq  = Cons(|0>, nq') }
}
qubitListToInt::(nq:List(Qubit) ; n:Int)=
{ case nq of
    Nil => { n = 0}
    Cons(q, nq') =>
          { n' = qubitListToInt(nq');
            measure q of
              |0> => {n1 = 0}
              |1> => {n1 = 1};
            use n1, n' in { n = n1 + (n' ≪ 1) }    }
}
```

Figure C.9: L-QPL code to convert integers from or to lists of qubits

The phase uses phase kickback to implement the U_0 transformation. Note this is an excellent example of the utility of both 0-based quantum control and quantum control by a data structure.

```
#Import Prelude.qpl

phase::(inqs:List(Qubit) ; outqs:List(Qubit))=
{  a=|1>;   Had a;
   Not a  <=  ~inqs; //Not a when all elts of inqs are |0>
   Had a;   Not a;   discard a;
   outqs=inqs
}
```

Figure C.10: L-QPL code using phase kickback to transform the list

The function doNGrovers calls the function gtrans repeatedly, based on the input n.

The gtrans function implements the G transform above. Note that we may ignore the minus sign in G's definition as we are using density matrices.

```
#Import oracle.qpl
#Import hadList.qpl
#Import phase.qpl

gtrans::(dataqbs:List(Qubit);
         dataqbs:List(Qubit))=
{  hadList dataqbs;
   phase dataqbs;
   hadList dataqbs;
   oracle dataqbs;
}


doNGrovers::(n:Int | dataqbs:List(Qubit);
                     dataqbs:List(Qubit))=
{  if (n==0) => {} //Done
   else      =>
      { gtrans dataqbs;
        doNGrovers(n-1) dataqbs}
}
```

Figure C.11: L-QPL code to do the grover transformation

The oracle for the grover algorithm is normally assumed to be a given. We provide a specific implementation where $f(12) = 1$.

```
#Import Prelude.qpl

oracle::(dataqbs:List(Qubit);
         dataqbs:List(Qubit))=
{  a = |1>;  Had a;
   case dataqbs of
     Nil =>
         { dataqbs = Nil;
           Had a; Not a; discard a}
     Cons(hd,tl) =>
         { case tl of
             Nil =>
                 { dataqbs = Nil;
                   Had a; Not a; discard a; discard hd;}
             Cons(hd',tl') =>
                 { Not a  <=  ~hd, ~hd', tl';  // Search for 0 0 1 1 (12)
                   dataqbs = Cons(hd,(Cons(hd',tl')));
                   Had a; Not a; discard a }
         }
}
```

Figure C.12: L-QPL code using phase kickback when $f(12) = 1$

C.2.2 Simon's algorithm

Simon's algorithm determines a global property of a given function, f, *provided it is guaranteed to follow some rules*. The function f, from n bits to n bits, must either be one-to-one, or when for any vectors of bits x, y with $f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$ then $(x_1 \oplus y_1, \ldots, x_n \oplus y_n) = (s_1, \ldots, s_n)$ where $\oplus$ is exclusive-or and the vector s is the same for any choices of x or y.

The L-QPL program to implement Simon's algorithm is in figure C.13 on the next page with supporting routines in figure C.14. The hadList function is defined in figure C.31 on page 170 and the oracle and "blackboxes" for this implementation in figure C.15 on page 156 and figure C.16.

For a specific function $f : \mathtt{bit}^n \to \mathtt{bit}^n$, the quantum portion of the algorithm returns a value r such that $r \cdot s = 0$. The classical remainder of the algorithm usually runs the quantum portion a number of times to obtain different values of r and then performs Guassian elimintion on the series of linear equations to determine s. We present only the quantum portion in this example.

The quantum portion first creates a list of length n of qubits initialized to $|0\rangle$. The Hadamard transform is applied to the list, followed by the oracle for f followed by another Hadamard. The qubits are then measured and the result is r.

For a complete description and analysis of the algorithm, see [Wat] or [NC00].

In figure C.13, the type ZorO represents bits. The function qubToBitList converts a list of qubits to elements of type ZorO by repeated measures.

```
#Import oracle.qpl
#Import hadList.qpl

main::()=
{  dqs=Cons(|0>,Cons(|0>,Cons(|0>,Nil)));
   hadList dqs;
   oracle dqs;
   hadList dqs;
   qubToBitList(dqs ; bits)
}

qdata ZorO = { Z | O }

qubToBitList::(inqubits:List(Qubit);
                         out:List(ZorO))=
{ case inqubits of
     Nil => { out = Nil}
     Cons(hd, tl) =>
             { outtl = qubToBitList(tl);
               measure hd of
                 |0> => {out = Cons(Z, outtl)}
                 |1> => {out = Cons(O, outtl)}}
}
```

Figure C.13: L-QPL code to do Simon's algorithm

The function ndestLength is a non-destructive measure of the length of a list. The function makeZeroQubitList creates a list of zeroed qubits when given a length.

```
#Import Prelude.qpl

ndestLength::(inLis:List(a);
              len:Int, outLis:List(a))=
{  case inLis of
     Nil => { len = 0; outLis = Nil}
     Cons(hd,tl) =>
            { ndestLength(tl; tlen, tail);
              use tlen in { len = 1 + tlen};
              outLis = Cons(hd, tail)}
}

makeZeroQubitList::(len:Int | ; outLis:List(Qubit))=
{  if (len =< 0) => { outLis = Nil}
   else          => { outListTail = makeZeroQubitList( len-1 |);
                      outLis = Cons(|0>,outListTail)}
}
```

Figure C.14: L-QPL list functions

The oracle for Simon's algorithm is one of the more complex in the examples given in this thesis. It makes use of the decomposition of $f : \mathtt{bit}^3 \to \mathtt{bit}^3$ into three functions, $f_i : \mathtt{bit}^3 \to \mathtt{bit}$ where f_i gives the i-th component of f. These are represented by the functions bbox1, bbox2 and bbox3.

The oracle function creates a list of ancilla qubits which are then used in each of the bbox$n$ functions, with the ancilla's being exclusive-or'ed with the results of the functions.

The code for the black boxes simply applies various combinations of controlled Nots to create the desired boolean functions. The code in figure C.16 only shows the first component. The code for the second and third components is similar.

```
#Import blackboxes.qpl

oracle::(dataqbs:List(Qubit);
         dataqbs:List(Qubit))=
{  ndestLength(dataqbs; len, dataqbs);
   use len;
   makeZeroQubitList(len | ; ancillas);
   bboxfRecurse(1,len) dataqbs ancillas;
   discard ancillas;
}

bboxfRecurse::(start:Int, len:Int | dqs:List(Qubit), anc:List(Qubit);
                                    dqs : List(Qubit), anc :List(Qubit))=
{  if (len < start) => { }
   else => { bbox(start) dqs anc;
             bboxfRecurse((start +1), len) dqs anc }
}

//Assumption of three qubits - customize for each function
bbox::(index:Int | dqs:List(Qubit), anc:List(Qubit);
                   dqs:List(Qubit), anc:List(Qubit))=
{  if (index == 1) => { bbox1 dqs anc}
      (index == 2) => { bbox2 dqs anc}
   else            => { bbox3 dqs anc}
}
```

Figure C.15: L-QPL code implementing oracle for Simon's algorithm

```
#Import ListUtils.qpl
// bb1 results in  1,0,0,1,0,1,1,0 when applied to combos of x,y,z
bbox1::(dqs:List(Qubit), anc:List(Qubit);
         dqs:List(Qubit), anc:List(Qubit))=
{   case anc of
      Nil => { anc = Nil}
      Cons(hAnc,atail) =>
        { case dqs of
             Nil => { anc = Cons(hAnc,atail);
                      dqs = Nil}
             Cons(x,tl1)=>
               { case tl1 of
                    Nil  =>{anc = Cons(hAnc,atail);
                            dqs = Cons(x,Nil)}
                    Cons(y,tl2)=>
                      { case tl2 of
                           Nil => {anc = Cons(hAnc,atail);
                                   dqs = Cons(x,(Cons(y,Nil)))}
                           Cons(z,tl3)=>
                             { // Now have x, y and z to work with.
                               cq = |0>; Not cq  <=  y,z;
                               Not cq  <=  ~y,~z;
                               Not cq  <=  x;
                               Not hAnc  <=  cq;  discard cq;
                               anc = Cons(hAnc,atail);
                               dqs = Cons(x,Cons(y,Cons(z,tl3)))  }
                      }
               }
        }
}
```

Figure C.16: L-QPL code implementing first black box for Simon's algorithm

C.3 Quantum arithmetic

This section provides examples of functions that will do arithmetic on quantum values. The primary source of these algorithms is [VBE95], although the algorithms and types used in modular arithmetic are not ones I have encountered yet in the literature.

C.3.1 Quantum adder

This section provides subroutines that perform *carry-save* arithmetic on qubits. The carry and sum routines in figure C.17 function as gates on four qubits and three qubits respectively.

```
carry::(c0:Qubit, a:Qubit, b:Qubit, c1:Qubit;
        c0:Qubit, a:Qubit, b:Qubit, c1:Qubit) =
{   Not c1  ⇐  b, a;
    Not b   ⇐  a;
    Not c1  ⇐  b, c0
}
sum::(c:Qubit, a:Qubit, b:Qubit;
      c:Qubit, a:Qubit, b:Qubit) =
{ Not b  ⇐  a;
  Not b  ⇐  c;
}
carryRev::(c0:Qubit, a:Qubit, b:Qubit, c1:Qubit;
           c0:Qubit, a:Qubit, b:Qubit, c1:Qubit) =
{   Not c1  ⇐  b, c0;
    Not b   ⇐  a;
    Not c1  ⇐  b,a
}
```

Figure C.17: L-QPL code to implement carry and sum gates

Additionally, the reverse of the carry is also defined in the same figure. In order to define a subtraction, which can be defined as the reverse of the add function, we would also need to define the reverse of the sum function.

The addition algorithm adds two lists of qubits and an input carried qubit. The first list is unchanged by the algorithm and the second list is changed to hold the sum of the lists, as shown in figure C.18.

```
#Import carrysave.qpl
adder::(c0:Qubit, asin:List(Qubit), bsin:List(Qubit);
        c0:Qubit, asout:List(Qubit), aplusbout:List(Qubit)) =
{    case asin of
      Nil => { asout = Nil; aplusbout = Nil}
      Cons(a, taila) =>
        { case bsin of
             Nil => { asout = Nil; aplusbout = Nil;}
             Cons(b, tailb) =>
              { c1 = |0>;
                carry c0 a b c1;
                case tailb of
                  Nil => { Not b <= a;
                           sum c0 a b;
                           tailb = Cons(c1,Nil)}
                  Cons( t, tlb') =>
                         { tailb = Cons(t,tlb');
                           adder c1 taila tailb;
                           carryRev c0 a b c1;
                           sum c0 a b;
                           discard c1};
                asout = Cons(a,taila);
                aplusbout = Cons(b,tailb)}}
}
```

Figure C.18: L-QPL code to add two lists of qubits

The program proceeds down the lists A and B of input qubits, first applying the carry to the input carried qubit, the heads of A and B and a new zeroed qubit, c_1. When the ends of the lists are reached, a controlled not and the sum are applied. The output $A+B$ list is then started with c_1. Otherwise, the program recurses, calling itself with c_1 and the tails of the input lists. When that returns, carry and sum are applied, the results are "Consed" to the existing tails of the lists, c_1 is discarded and the program returns.

C.3.2 Modular arithmetic

Most treatments of quantum modular arithmetic assume the program / algorithm will work with a quantum register with a sufficiently large number of qubits, as in

[VBE95]. In L-QPL, we define a new datatype for quantum modular integers, called QuintMod. A QuintMod consists of a triple of two integers and a list of qubits. The first integer is the maximum size of the list and the second integer is the modulus.

```
#Import ModFunctions.qpl
#Import ListFunctions.qpl
#Import quints.qpl
qdata QuintMod = {QuintMod(Int,Int,List(Qubit))}

//Convert a probabalistic int to a QuintMod
intToQuintMod::(radix:Int, n:Int ;  nq:QuintMod)=
{  use n, radix;
   determineIntSize(radix;  size);
   cintToQuintMod(n, radix | size ; nq)
}

//Convert a classical int to a QuintMod
cintToQuintMod::(n:Int, radix:Int | size:Int ; nq:QuintMod)=
{ if n == 0 => { nq = QuintMod(size,radix,Nil) }
  else      => { nmr  := n mod radix;
                 qlist = intToQubitList(nmr | );
                 nq    = QuintMod(size,radix,qlist) }
}

//Convert a QuintMod to a probabalistic integer
quintModToInt::(nq:QuintMod ; n:Int)=
{  case nq of
     QuintMod(_,radix,digits) =>
       { n' := qubitListToInt(digits);
         use radix in {n  = n' mod radix}}
}
```

Figure C.19: L-QPL definitions of QuintMod

The code for the type definition and conversion from and to integers is given in figure C.19.

Given these definitions and code for adding and subtracting lists of qubits(as in appendix C.3.1), it is now possible to define a *smart constructor* for a QuintMod.

A QuintMod triplet has only one invariant that must be maintained. That is the list of qubits must have length less than or equal to the first number of the triplet. Note this implies that the number represented by a qubit list may be outside the range of the modulus. For example, assume a modulus of 5. This implies a length of 3 qubits. The (qu)bit sequences of 101, 011, 111, representing 5, 6 and 7 are allowed. When converting back to a viewable integer, these would be converted to 0, 1, 2 respectively.

```
#Import Prelude.qpl
#Import carrysave.qpl
#Import QuintMods.qpl

// Needs to subtract the modulus 0,1 or 2 times to
// get the correct range for the numbers.
makeQuint::(size:Int, radix:Int, digits:List(Qubit) ;
                                   res:QuintMod)=
{  getLength(digits; digits,length);
   use size,length in
     { if length > size =>
         { // Two controlled subs
           use radix;
           csubModulus(radix | digits ; digits);
           takeOnly(size+1 | digits ; digits); //Last is |0>
           csubModulus(radix | digits ; digits);
           takeOnly(size | digits ; digits); //Last is |0>
           res = QuintMod(size,radix,digits) }
       else =>
         { res = QuintMod(size,radix,digits)}
     }
}

csubModulus::(modulus:Int | digits:List(Qubit) ;
                           digs:List(Qubit))=
{  splitLast(digits, |0> ; digits, last);
   ctldintToQubitList(modulus | last ; last, subRad);
   digs = append(digits,Cons(last,Nil));
   c0   = |0>;
   subLists c0 subRad digs;
   discard c0, subRad;
}
```

Figure C.20: Semi-Smart constructor for QuintMod

The function makeQuint defined in figure C.20 maintains the length invariant, *assuming* that the length is no more than 1 greater than allowed. That is, if working with QuintMod numbers of length n, the List(**Qubit**) argument has length at most $n+1$.

This will imply the represented number passed in is at most $2^{n+1}-1$. As the modulus is at least 2^{n-1}, at most two subtractions of the modulus will ensure the passed in list represents a number in the range $0 - 2^n - 1$. The subtractions are controlled by the $n+1$st qubit in the list.

The reason the numbers may take on the full range is due to the representation. For example, if we are working in mod 4, which requires 3 qubits for representation we may attempt to add 7 to itself. This will result in 14, requiring 4 qubits. The final qubit, recalling we store numbers with least significant qubit first, will be $|1\rangle$. Subtracting 4 once leaves us with 10, still requiring 4 qubits. Subtracting 4 once more gives us 6 which brings us back to the correct range.

```
#Import smartConstructor.qpl

addM::(asin:QuintMod, bsin:QuintMod ;
      asout:QuintMod, aplusbout:QuintMod) =
{ case asin of QuintMod(sizeA,mA, aDigits) => {
  case bsin of QuintMod(sizeB,mB, bDigits) =>
    { use sizeB;
      normalize( sizeB| bDigits;bDigits);
      c0 = |0>;
      addLists c0 aDigits  bDigits;
      discard c0;
      asout     = QuintMod(sizeA,mA, aDigits);
      aplusbout = makeQuint(sizeB,mB, bDigits)}}
}

addListToQuint::(inc:List(Qubit), dest:QuintMod;
                 dest:QuintMod)=
{ case dest of QuintMod(sizeD,mdD, dDigits) =>
    { use sizeD,mdD;
      a    = QuintMod(sizeD,mdD, inc);
      dest = QuintMod(sizeD,mdD, dDigits);
      addM a dest;
      discard a}
}
```

Figure C.21: Quantum modular addition

Figure C.21 shows the L-QPL code for implementing modular addition. The function addM takes in two QuintMod elements and adds the first to the second. Note the use of the smart constructor makeQuint to produce the result.

```
#Import basicArith.qpl

ctlCopy::( ctl:Qubit, src:QuintMod ;
           ctl:Qubit, src:QuintMod, dest:QuintMod)=
{   case src of QuintMod(size,modulus,srcDig)=>
     { use size, modulus;
       ctlCopyList(ctl, srcDig; ctl, srcDig, destDig);
       dest = QuintMod(size,modulus,destDig);
       src  = QuintMod(size,modulus,srcDig)}
}

ctlDouble::( ctl:Qubit, src:QuintMod ;
             ctl:Qubit, src:QuintMod, dest:QuintMod)=
{   case src of QuintMod(size,modulus,srcDig)=>
     { use size, modulus;
       ctlCopyList(ctl, srcDig; ctl, srcDig, destDig');
       destDig = Cons(|0>, destDig');
       dest    = makeQuint(size,modulus,destDig);
       src     = QuintMod(size,modulus,srcDig)}
}

ctlCopyList::(ctl:Qubit, srcList:List(Qubit);
              ctl:Qubit, srcList:List(Qubit), destList:List(Qubit))=
{ case srcList of
    Nil => { srcList = Nil; destList = Nil}
    Cons(hd,tl) =>
          { ctlCopyList(ctl, tl; ctl, tl, desttl);
            desthd   = |0>;
            Not desthd <= ctl, hd;
            srcList  = Cons(hd,tl);
            destList = Cons(desthd,desttl)}
}
```

Figure C.22: Support functions for Quantum modular multiplication

At this stage modular multiplication is straightforward to write. First, the support functions ctlCopy and ctlDouble are defined in figure C.22. The ctlCopy function creates a "copy" of a QuintMod. Note that the qubitsare all created by controlled-Nots of the original list, so the "copy" is entangled with the original. If the control qubit is $|0\rangle$, the resulting QuintMod is zero. The ctlDouble function uses the controlled copying and introduces a new $|0\rangle$ at the beginning of the list of qubits in the QuintMod. This effectively doubles the number by shifting it one position to the right. The resulting QuintMod is created using the smart constructor defined previously.

```
#Import multSupport.qpl

multiplyM::(cor:QuintMod, cand:QuintMod ;
            cor:QuintMod,  res:QuintMod)=
{  case cand of QuintMod(sizeA,modA,aDig) =>
     { use sizeA,modA;
       res = cintToQuintMod(0,modA | sizeA);
       case aDig of Nil => {}
         Cons(hdA,tlA)=>
           { ctlCopy(hdA,cor;hdA,cor,corcopy);
             discard hdA;
             addM corcopy res;
             discard corcopy;
             ctl = |1>;
             ctlDouble(ctl,cor; ctl, cor, cordouble);
             discard ctl;
             res1  = QuintMod(sizeA, modA, tlA);
             multiplyM  cordouble res1; discard cordouble;
             addM res1 res; discard res1}
     }
}
```

Figure C.23: Quantum modular multiplication

The function multiplyM is then defined as a recursive function in figure C.23. The algorithm used is the standard grade school multiplication method.

Modular exponentiation is written in a similar manner. First the support functions ctlCopyOne and square are defined in figure C.24. ctlCopyOne uses ctlCopy defined above to create a copy of a QuintMod, but adds 1 to the resulting QuintMod when the control qubit is $|0\rangle$, resulting in the identity for multiplication rather than the identity for addition. The function square corresponds to ctlDouble in the multiplication case.

The code for the actual powerM function is in figure C.25.

```
#Import multiply.qpl

ctlCopyOne::(ctl:Qubit, src:QuintMod ;
             ctl:Qubit, src:QuintMod, dest:QuintMod)=
{ ctlCopy(ctl,src; ctl,src,dest); //dest == 0 if ctl == 0
  ctlone = |0>;
  Not ctlone <= ~ctl;
  addListToQuint(Cons(ctlone,Nil), dest; dest);
}

square::( src:QuintMod; dest:QuintMod)=
{ ctl = |1>;
  ctlCopy(ctl, src; ctl,dest,cpy);  discard ctl;
  multiplyM cpy dest;  discard cpy
}
```

Figure C.24: Quantum modular exponentiation support

```
#Import powerSupport.qpl

powerM::(base:QuintMod, pow:QuintMod ;
                        res:QuintMod)=
{  case pow of QuintMod(sizeA,modA,aDig) =>
     { use sizeA,modA;
       res = cintToQuintMod(1,modA|sizeA);
       case aDig of Nil => {discard base} //  base^0 is 1
       Cons(hdA,tlA)=>
         { ctlCopyOne(hdA,base;hdA,base,basecopy);
           discard hdA;
           multiplyM basecopy  res;  discard basecopy;
           square(base; basesqd);
           pow' = QuintMod(sizeA, modA, tlA);
           powerM(basesqd,pow'; res1);
           multiplyM res1 res;  discard res1}}
}
```

Figure C.25: Quantum modular exponentiation

C.4 Order finding

Shor's algorithm for factoring depends on the quantum algorithm for order finding. Shor's algorithm may be summarized by the following three steps:

- Setup;
- Call order finding (a quantum algorithm);
- Check to see if an answer has been found, repeat if not.

In this section, we concentrate on the quantum portion, order finding. Theoretical details may be found in [NC00].

The code for the main order finding function, OrderFind is shown in figure C.26

```
1  #Import powerFind.qpl
2  #Import inverseQft.qpl
3
4  orderFind::(n:Int, x:Int | ; order:Int)=
5  {
6     size    := determineIntSize(n);
7     sizeT   := 2 × size + 1 + 2;
8     makeZeroQubitList(sizeT | ; tList );
9     hadList tList;
10    xQm      = intToQuintMod(x);
11    powList xQm tList;
12    discard xQm;
13    inverseQft tList;
14    result   = qubitListToInt(tList);
15    gcdrs   := gcd(result,(2 ≪ sizeT));
16    order    = sizeT mod gcdrs;
17 }
```

Figure C.26: L-QPL function for order finding.

Beginning at lines 1 and 2, the program imports files which provide various other functions. At line 4, the function is declared to take two classical integers as input and return one probabalistic integer. The first two lines of code, 6 and 7 compute the number of qubits required for the algorithm to work with a maximum error of $\frac{1}{4}$. Increasing sizeT will reduce the error.

Lines 8 through 10 complete the initial setup for the algorithm. These include: Creating tList, a list of sizeT qubits initialized to zero; applying the Hadamard transform to each qubit; creating xQm, a QuintMod with the value of the input parameter x. See appendix C.3.2 on page 159 for the details of modular arithmetic.

Line 11 performs the modular exponentiation and is shown in figure C.27. This is similar to the exponentiation function in figure C.25 on page 165 and has just been modified to work with an exponent in list form and to retain the exponent.

```
#Import power.qpl

powList::(base:QuintMod, pow:List(Qubit) ;
          res:QuintMod,  pow:List(Qubit))=
{  case base of QuintMod(sizeA,modA,baseDig) =>
     { use sizeA,modA;
       res = cintToQuintMod(1,modA|sizeA);
       base = QuintMod(sizeA,modA,baseDig);
       case pow of
          Nil          => {pow=Nil; discard base} //  base^0 is 1
          Cons(hdA,tlA) =>
            { ctlCopyOne(hdA,base;hdA,base,basecopy);
              multiplyM basecopy  res;
              discard basecopy;
              square(base; basesqd);
              powList basesqd tlA;
              multiplyM basesqd res;  discard basesqd;
              pow = Cons(hdA,tlA)}}
}
```

Figure C.27: L-QPL code for modular power by a list

On completion of the exponentiation, the actual result is discarded as it has no further effect on the algorithm. Then, the inverse quantum Fourier transform (shown in figure C.28 and figure C.29) is applied to tList and it is measured to produce a probabalistic integer in line 14.

```
#Import inverseRotate.qpl
#Import Utils.qpl

inverseQft :: (inqs:List (Qubit); outqs:List (Qubit)) =
{  reverse inqs;
   inverseQft' inqs;
   outqs = reverse(inqs);
}

inverseQft' :: (inqs:List (Qubit); outqs:List (Qubit)) =
{ case inqs of
    Nil => {outqs = Nil}
    Cons(h,inqs') =>
      {  inverseQft' inqs';
         inverseRotate (2) h inqs';
         Had h;
         outqs = Cons(h,inqs') }
}
```

Figure C.28: Function to apply the inverse quantum Fourier transform

```
#Import Prelude.qpl

inverseRotate ::(n:Int| h:Qubit, inqs:List (Qubit);
                 h:Qubit, outqs:List (Qubit))=
{ case inqs of
    Nil => {outqs = Nil }
    Cons (q, inqs') =>
      { use n;
        m := n+1 ;
        inverseRotate(m) q inqs';
        Inv-Rot(n) h <= q;
        outqs = Cons(q,inqs') }
}
```

Figure C.29: Function to apply inverse rotations as part of the inverse QFT

The final part of the algorithm is normally described as using a continued fraction algorithm to compute the potential order. This is the same as dividing 2^{sizeT} by the greatest common division of itself and the result.

At this point, since this is a probabilistic algorithm, the calling function would need to check whether the result is correct. If so, it may then be used in the completion of the factoring algorithm.

Various other support functions for this algorithm are shown in figure C.30 and figure C.31.

```
#Import inverseQft.qpl
#Import ListUtils.qpl

gcd::(a:Int, b:Int ; ans: Int) =
{  use a, b in {
     if b == 0 => {ans = a}
        a == 0 => {ans = b}
        a >= b => {ans = gcd(b, a mod b)}
     else      => {ans = gcd(a, b mod a)}}
}
```

Figure C.30: GCD and import of other ulitlties

```
#Import Prelude.qpl

ndestLength::(inLis:List(a);
               len:Int, outLis:List(a))=
{  case inLis of
     Nil => { len = 0; outLis = Nil}
     Cons(hd,tl) =>
            { ndestLength(tl; tlen, tail);
              use tlen in { len = 1 + tlen};
              outLis = Cons(hd, tail)}
}

makeZeroQubitList::(len:Int | ; outLis:List(Qubit))=
{  if (len =< 0) => { outLis = Nil}
   else          => { outListTail = makeZeroQubitList( len-1 |);
                      outLis = Cons(|0>,outListTail)}
}


hadList::(inhqs:List(Qubit) ; outhqs:List (Qubit))=
{ case inhqs of
    Nil             => { outhqs = Nil }
    Cons(q,hadtail) =>
       {  Had q;
          hadList hadtail;
          outhqs = Cons(q, hadtail)}
}

append::(list1:List(a), list2:List(a) ; app:List(a))=
{ case list1 of
     Nil            => {app = list2}
     Cons(a,subl1) =>
       { app = Cons(a, append(subl1, list2)) }
}

reverse::(inlis:List(a) ; rvlis:List(a))=
{  rvlis = rev'(inlis, Nil) }

rev'::(inlist:List(a), accin:List(a) ; revlist:List(a)) =
{ case inlist of
    Nil             => {revlist = accin}
    Cons(a,sublist) =>
      { acc     = Cons(a, accin);
        revlist = rev'(sublist, acc) }
}
```

Figure C.31: Various list ulitlties

Appendix D

Using the system

D.1 Running the L-QPL compiler

The compiler is run from the command line as:

```
lqplc <options> <infiles>
```

This will run the compiler on each of the input files *infiles* which are expected to have a suffix `'.qpl'`. The compiled files will be written with the suffix `'.qpo'`.

The options allowed for the compiler are:

`-e, --echo_code` Echo the input files to `stderr`.

`-s, --syntactic` This option will cause the compiler to print a syntax parse tree on `stderr`.

`-r, --ir_print` This compiler option will force the printing of the intermediate representation generated during the semantic analysis phase.

`-h, --help` This prints a help message describing these options on `stderr`.

`-V, -?, --version` This option prints the version information of the compiler on `stderr`.

`-o[FILE], --output[=FILE]` This will cause the compiler to write the compiled QSM code to `FILE`.

`-i[DIRLIST], --includes=DIRLIST` For this option, `DIRLIST` is expected to be a list of semi-colon separated directories. The compiler will use the directory list when searching for any import files.

D.2 Running the QSM Emulator

D.2.1 Window layout

When starting the simulator with the command `qsmemul`, The first thing seen is figure D.1

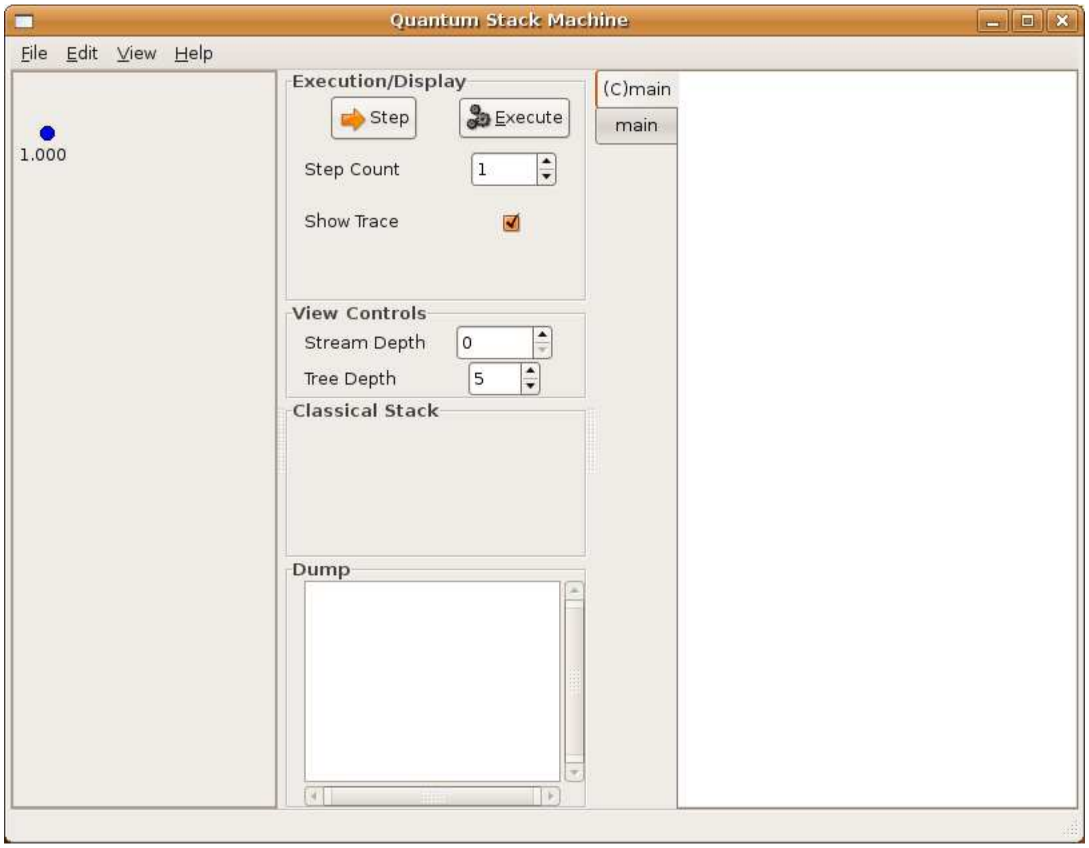


Figure D.1: The emulator window

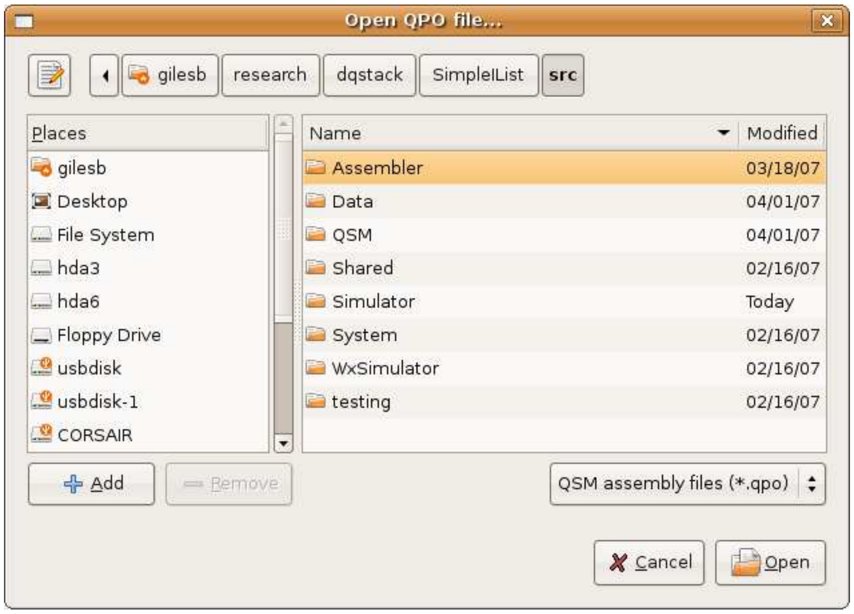
The emulator window is composed of three main sections. On the left is the quantum stack display area. On the right is a tabbed display of the emulator assembly/-

machine code.

The middle section is divided into four parts. At the top is an execution and display control area followed by a view control area. Beneath these is the classical stack display area with the dump display area at the bottom.

D.2.2 Loading a file

The first required step is to actually load a program into the emulator, using the `File --> Open` dialog in figure D.2.



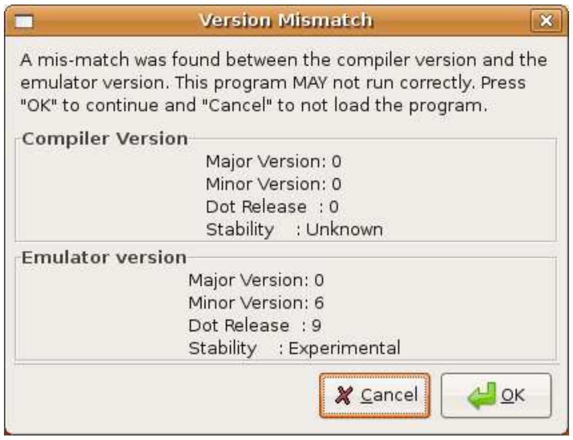
Name	Modified
Assembler	03/18/07
Data	04/01/07
QSM	04/01/07
Shared	02/16/07
Simulator	Today
System	02/16/07
WxSimulator	02/16/07
testing	02/16/07

QSM assembly files (*.qpo)

Figure D.2: Emulator file open dialog

Opening a QSM assembly file will check the compiler version used to create it and translate it to machine code. If the compiler and emulator versions do not match, a warning dialog, as in figure D.3 will appear. It is suggested that one ensures the version of the compiler and emulator are the same.

After a successful assemble, the assembled code will appear in the right hand side. In a program, each function will appear as a separate tab in the tabbed code window.


Version Mismatch

A mis-match was found between the compiler version and the emulator version. This program MAY not run correctly. Press "OK" to continue and "Cancel" to not load the program.

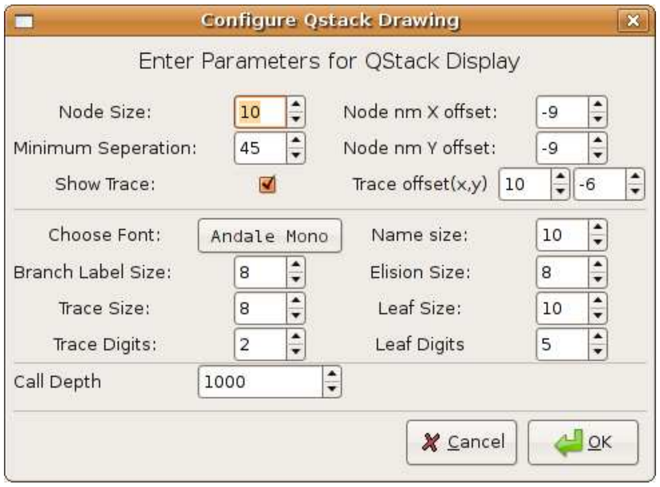
Compiler Version
Major Version: 0
Minor Version: 0
Dot Release : 0
Stability : Unknown

Emulator version
Major Version: 0
Minor Version: 6
Dot Release : 9
Stability : Experimental

Figure D.3: Version mis-match warning

The currently executing code will be in the top tab, which is labelled with a "(C)" and the name of the currently executing function. Clicking on a tab will show the code associated with that tab.

D.2.3 Setting preferences


Configure Qstack Drawing

Enter Parameters for QStack Display

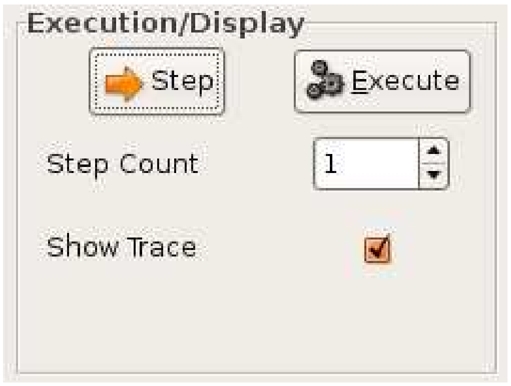
Node Size: 10
Minimum Seperation: 45
Show Trace: ✓
Node nm X offset: -9
Node nm Y offset: -9
Trace offset(x,y) 10 -6
Choose Font: Andale Mono
Branch Label Size: 8
Trace Size: 8
Trace Digits: 2
Name size: 10
Elision Size: 8
Leaf Size: 10
Leaf Digits 5
Call Depth 1000

Figure D.4: Preferences dialog

Prior to executing the program, various display and execution options may be set by using the `Edit --> Preferences` dialog, shown in figure D.4. The top third of the dialog controls the spacing and size of nodes and their labels. The middle controls the font used and the font size choices for various labels. Typically, the defaults for these are sufficient for general execution.

The final section, with the single item `Call Depth` is used to control the actual depth of recursion *multiplied by the depth in the stream.* In figure D.4 the call depth is 1000, meaning that at stream depth 0, we will perform 1000 calls before signalling non-termination, at stream depth 5, 6000 calls and so on.

D.2.4 Running the program



Step Count 1

Show Trace ✓

Figure D.5: Execution control section of the main window

The program execution is controlled by the elements of the `Execution/Display` section of the main window. As shown in figure D.5, there are two buttons (`Step` and `Execute`), a spin control (`Step Count`) and a checkbox (`Show Trace`).

For each click of the `Step` button, the emulator will execute *`Step Count`* instructions and then redisplay the components of the quantum stack machine. The spin control may be set to any positive number.

The `Execute` button will run the program until it completes. Completion may either be due to non-termination (e.g., exceeding the call depth number of calls at the current stream depth), or actual completion. See the description of `Stream Depth` below. As well, while executing, the program will display a progress bar below the `Show Trace` area. After completion, the machine components will be re-displayed.

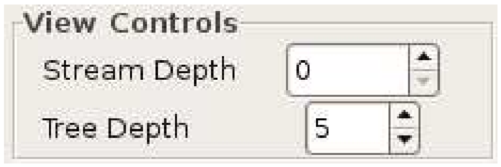


Figure D.6: View controls section of the main window

The other component that affects execution in both step mode and execute mode is the `Stream Depth`, shown in figure D.6. Essentially, if one encounters a non-termination (a zero quantum stack) after executing, just increase the `Stream Depth` and continue.

D.2.5 Result interpretation

Obviously, the first step is to visually examine the quantum stack. In case where a simple final result is produced, this is often enough.

However, in cases where there are a significant number of nodes with multiple branches, it can be difficult to determine what the results actually are.

For example, consider the end result of running Simon's algorithm, as shown in figure D.7. The important result is "What are the resulting bit-strings?"[1]. Using the menu item `File --> Simulate`, we bring up a simulation dialog, which does the "roll the dice" and shows us what our end result would be when transferred to a classical computer. For example, for three invocation of simulation, we get sub-figures (a), (b), and (c) as shown in figure D.8.

Note that the bit string can simply be read from the top three entries in the simulation results. It should also be noted that the additional simulations do not require

[1]Obviously, it is possible to determine the results solely by examination of the quantum stack as displayed. The method shown here becomes more relevant the more complicated the result stack.

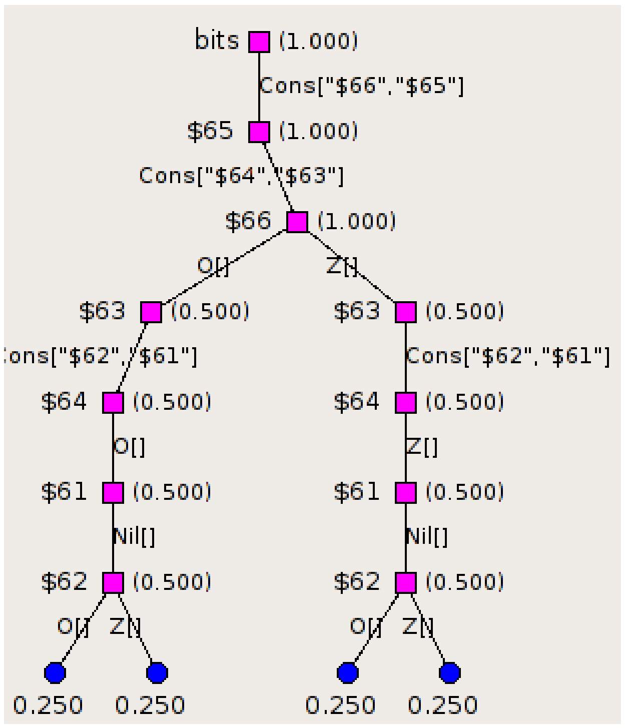
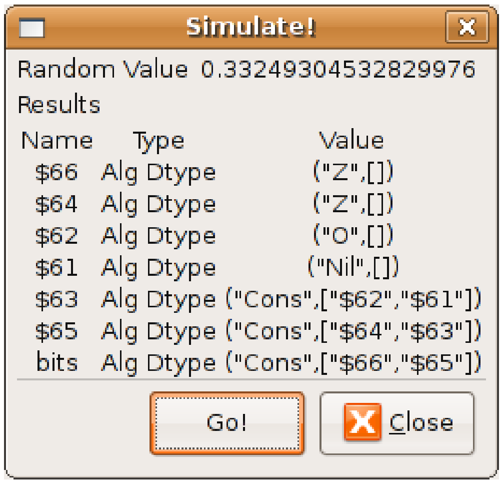


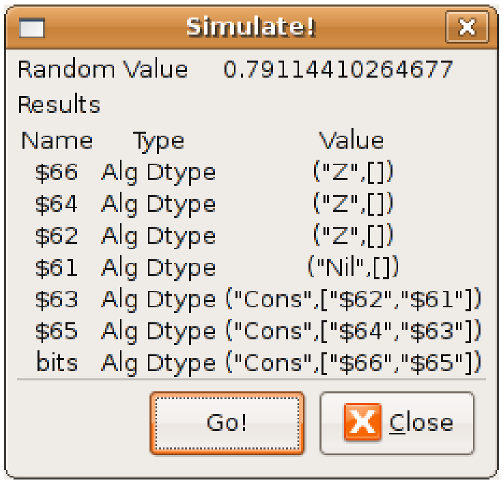
Figure D.7: Quantum stack at end of Simon's algorithm

re-execution by the emulator. Instead, a new random value is generated and used to determine a single path down the quantum stack for the values.



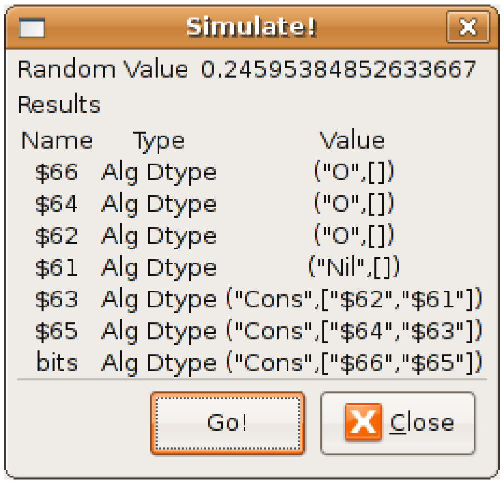
```
Random Value 0.33249304532829976
Results
Name   Type        Value
 $66   Alg Dtype        ("Z",[])
 $64   Alg Dtype        ("Z",[])
 $62   Alg Dtype        ("O",[])
 $61   Alg Dtype       ("Nil",[])
 $63   Alg Dtype ("Cons",["$62","$61"])
 $65   Alg Dtype ("Cons",["$64","$63"])
 bits  Alg Dtype ("Cons",["$66","$65"])
```

(a) First simulation



```
Random Value    0.79114410264677
Results
Name   Type        Value
 $66   Alg Dtype        ("Z",[])
 $64   Alg Dtype        ("Z",[])
 $62   Alg Dtype        ("Z",[])
 $61   Alg Dtype       ("Nil",[])
 $63   Alg Dtype ("Cons",["$62","$61"])
 $65   Alg Dtype ("Cons",["$64","$63"])
 bits  Alg Dtype ("Cons",["$66","$65"])
```

(b) Second simulation



```
Random Value 0.24595384852633667
Results
Name   Type        Value
 $66   Alg Dtype        ("O",[])
 $64   Alg Dtype        ("O",[])
 $62   Alg Dtype        ("O",[])
 $61   Alg Dtype       ("Nil",[])
 $63   Alg Dtype ("Cons",["$62","$61"])
 $65   Alg Dtype ("Cons",["$64","$63"])
 bits  Alg Dtype ("Cons",["$66","$65"])
```

(c) Third simulation

Figure D.8: Simulation of Simon's algorithm